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String theory in electromagnetic fields

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Abstract: A review of various aspects of superstrings in background electromagnetic fields is presented. Topics covered include the Born-Infeld action, spectrum of open strings in background gauge fields, the Schwinger mechanism, finite-temperature formalism and Hagedorn behaviour in external fields, Debye screening, D-brane scattering, thermodynamics of D-branes, and noncommutative field and string theories on D-branes. The electric field instabilities are emphasized throughout and contrasted with the case of magnetic fields. A new derivation of the velocity-dependent potential between moving D-branes is presented, as is a new result for the velocity corrections to the one-loop thermal effective potential.

Keywords: Superstrings and Heterotic Strings, D-branes, M(atrix) Theories.

Contents

1. Introduction

Superstring theory is a fundamental candidate for a theory of quantum gravity because its elementary closed string spectrum naturally induces background fields of ten-dimensional supergravity. Among the bosonic fields one finds, in addition to the metric tensor of ten-dimensional spacetime, a torsion Neveu-Schwarz two-form field as well as higher and lower degree differential form Ramond-Ramond fields. The former field, when it is closed or equivalently on-shell, is formally equivalent to a background electromagnetic field strength tensor in spacetime, while the latter ones are the objects which couple to D-branes, the extended hyperplanes in spacetime onto which open strings attach (with Dirichlet boundary conditions). Dp-branes are p-dimensional soliton-like objects whose quantum dynamics are described by the quantum theory of the open strings whose ends are constrained to move on them. In the low-energy limit, there are a finite number of massless fields which survive whose dynamics are described by a $p+1$-dimensional effective quantum field theory. One of these fields is a U(1) gauge field. Therefore, understanding the behaviour of strings and D-branes in the presence of electromagnetic fields is important for the description of non-perturbative vacuum states in superstring theory. Furthermore, using duality, this problem is also important for understanding various aspects of D-brane dynamics.

In this review article we shall focus on a particularly tractible problem, that of open strings and D-branes in constant background electric and magnetic fields. These models have attracted renewed interest very recently because they give an explicit realization of some old conjectures about the nature of spacetime at very short distance scales. If one is to use string states as probes of short distance structure, then one cannot probe lengths smaller than the intrinsic length of the strings. Therefore, below the string scale the notion of geometry must drastically change, and an old proposal is that the spacetime coordinates become noncommuting operators. The deformation of D-brane worldvolumes to noncommutative manifolds by the external electromagnetic field has led to a revival of interest in these earlier suggestions. In addition, the effective low-energy dynamics can be described by new, noncommutative versions of ordinary quantum field and string theories, and hence a wealth of new problems for both field theorists and string theorists.

Motivated by these issues, in the following we will present an overview of some of the fundamental aspects of string theory in electromagnetic fields. The qualitative effects can all be seen at the level of the simpler bosonic string theory, which we will confine most of our attention to in this paper. As we indicate throughout, the results readily extend to the case of superstrings. Many of the novel effects exhibited by strings in background fields can be seen at the level of free open strings, or equivalently (in type-IIB superstring theory) for D9-branes which fill the spacetime. This is the topic of section 2. We will derive the effective gauge field dynamics for the open strings up to one-loop order in string perturbation theory, and describe the spectrum of the string theory. We shall also start seeing here some important differences between electric and magnetic backgrounds in superstring theory. While strings in external magnetic fields possess no more instabilities than the quanta of Yang-Mills gauge theory, electric backgrounds play a much different role in string theory. In addition to the usual instability of the vacuum in an electric background that occurs in quantum electrodynamics, strong electric fields can tear apart a string and render both the classical and quantum theories physically meaningless.

As we have mentioned, string theory exhibits a variety of novel effects at very high energies. Non-trivial background fields may also have an effect on the properties of strings in this regime. In particular, one can examine how the external fields modify the behaviour of strings at high temperatures, where they are known to undergo a phase transition into a sort of deconfining phase in which the strings propagate as long string states in the spacetime. Free strings at finite temperature and in background electromagnetic fields will be analysed in section 3.

One of the most important applications of the external field problem for free open strings is its interpretation in the T-dual picture, where it maps onto the problem of moving D-branes. This problem is dealt with at length in section 4. Here we present a new derivation, which contains some novel technical details that may be of use for other calculations, of the well-known scattering amplitude between two D-branes travelling at constant velocity. The corresponding thermodynamic problem is particularly interesting in this case. A special class of black holes in string theory admits a dual description as a configuration of D-branes. By using the quantum string theory living on the D-brane, one can compute the Bekenstein-Hawking entropy and the rate of thermal radiation from the black hole. The corresponding Hawking temperature is conjectured to be the same in this case as the extrinsic temperature of a Boltzmann gas of D-branes. These features of the thermal ensemble of D-branes can be checked by computing the free energy using the effective, low energy description of D-brane dynamics in terms of supersymmetric Yang-Mills theory with 16 supercharges. In section 4 we shall also present new results for the leading velocity corrections to the one-loop thermal potential between D-branes.

The final instance of the constant external field problem, which we address in section 5, is to study the properties of D-branes themselves in the electromagnetic background. Here we shall focus on the geometric modifications that are caused by the external field. We shall see that, generically, the D-brane worldvolume is not a conventional manifold and is described by a noncommutative space. This is again a particular effect of the quantum open string theory that lives on the D-branes. Here we shall see a particularly drastic distinction between electric and magnetic fields. In a particular low-energy limit, the effective dynamics of the noncommutative D-branes is described in the magnetic case by a deformation of the usual gauge field dynamics on the branes, while in the electric case there is no field theory limit and the effective theory is a deformation of the usual open string theory on the D-branes. In this latter case, the noncommutativity is given directly in terms of the string scale, and the most intesting aspect of this open string theory on the noncommutative manifold is that it does not contain closed strings. In particular, it is a novel example of a string theory which does not contain gravity. We can expect that these theories capture many of the important features of the standard string theories, but without being plagued by the conceptual problems that arise due to the presence of gravitation.

2. Open strings in background gauge fields

In this section we will start describing some of the basic physical properties of strings in an external electromagnetic field. An external gauge field couples to an open string through Chan-Paton factors at the string endpoints. Therefore, because of the Green-Schwarz anomaly cancellation condition, all of our considerations in this section and the next strictly speaking only apply to type-I superstrings, since type-II superstring theory has no gauge group. The gauge field is then associated with a subgroup of the $\mathrm{SO}(N)$ gauge group of type-I string theory, where $N = 2^{d/2}$ and d is the dimension of spacetime which we will assume is even. By an electromagnetic field we will mean one that is associated with an abelian subgroup of this gauge group. However, we will only write down explicit formulas which also pertain in principle to type-II superstrings, as they will become relevant in sections 4 and 5 when the open strings will attach to D-branes which can host electromagnetic fields in the guise of a Neveu-Schwarz two-form field.

2.1 The Born-Infeld action

In this subsection we will derive the low-energy effective action which governs the propagation of free open strings in a slowly-varying background electromagnetic field $F_{\mu\nu}$ [25, 1, 16] (See [57] for a review). In string perturbation theory and in the RNS formulation, the vacuum energy may be computed in first quantization from the Polyakov path integral

$$\mathcal{Z}[F] = -\sum_{h=0}^{\infty} g_s^{2h-1} \sum_{\sigma} \int Dg_{ab}\, D\chi_{ab} \int Dx^\mu\ D\psi^\mu\ \mathrm{e}^{\,-S[g,x,\chi,\psi;A]}\ , \tag{2.1}$$

where g_s is the string coupling constant whose powers weight the genus h of the open string worldsheet which has euclidean metric g_{ab} (and superpartner the two-dimensional gravitino field χ_{ab}), and the sum over spin structures σ with the appropriate weights imposes the GSO projection that leads to modular invariance, a tachyon-free spectrum, and spacetime supersymmetry of the string theory. We will assume in this section that the target space has flat euclidean metric $\delta_{\mu\nu}$.

The first contribution to the partition function (2.1) comes from the disc diagram Σ, which by conformal invariance of the classical theory can be parametrized by coordinates $z = r\,\mathrm{e}^{\,i\vartheta}$ with $0 \leq r \leq 1$ and $0 \leq \vartheta \leq 2\pi$. The bosonic part of the tree level string action in the conformal gauge is

$$S_0[x,A] = \frac{1}{4\pi\alpha'} \int d^2z\ \partial x^\mu\, \overline{\partial} x_\mu - ie \int_0^{2\pi} d\vartheta\ \dot{x}^\mu\, A_\mu(x)\Big|^{r=1}\ , \tag{2.2}$$

where here and in the following a dot will denote differentiation with respect to the worldsheet boundary coordinate ϑ. The quantity $T_s = 1/2\pi\alpha'$ is the string tension. The endpoints of the open strings carry charges e which couple to the electromagnetic vector potential $A_\mu(x)$. When inserted into (2.1), the action (2.2) leads to the expectation value, with respect to the free worldsheet σ-model (the bulk term in (2.2)), of the Wilson loop operator for the gauge field A_μ over the boundary of Σ. To evaluate it, we use the background field approach and expand the string embedding fields as $x^\mu = x_0^\mu + \xi^\mu$, where x_0^μ are their constant zero modes. The tree-level contribution to (2.1) then involves the propagator $G^{\mu\nu}(z,z') = \langle 0|T[\xi^\mu(z)\xi^\nu(z')]|0\rangle = -2\pi\alpha'\,\delta^{\mu\nu}\,N(z,z')$, where

$$N(z,z') = \frac{1}{2\pi}\,\ln\left|z-z'\right|\left|z-\overline{z}'^{\,-1}\right| \tag{2.3}$$

is the Neumann function for the disc which satisfies the equation of motion $\nabla^2 N(z,z') = \delta(z-z')$ and the Neumann boundary condition $\partial_r N(z,z')|_{r=1} = 0$.[1] On the boundary of the worldsheet, where $z = \mathrm{e}^{\,i\vartheta}$, the Green's function (2.3) becomes

$$\begin{aligned} N(\vartheta,\vartheta') &= \frac{1}{2\pi}\,\ln\Big(2 - 2\cos(\vartheta-\vartheta')\Big) \\ &= -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos n(\vartheta-\vartheta')}{n}\,\mathrm{e}^{\,-\varepsilon n}\ , \end{aligned} \tag{2.4}$$

[1] This Green's function may be derived by using the method of images after mapping the disc into the upper complex half-plane via a conformal transformation.

where $\varepsilon \to 0^+$ is an ultraviolet cutoff which regulates the logarithmic short-distance $\vartheta \to \vartheta'$ singularity of (2.4), $N(\vartheta,\vartheta) = \frac{1}{\pi}\ln\varepsilon$. We will work in the radial gauge $\xi^\mu A_\mu(x_0+\xi) = 0$, $A_\mu(x_0) = 0$, and with slowly varying gauge fields which admit an expansion $A_\mu(x_0+\xi) = \frac{1}{2}F_{\mu\nu}(x_0)\,\xi^\nu + \mathcal{O}(\partial F)$. In the following we will evaluate the vacuum amplitude to leading orders in the expansion in derivatives of the field strength tensor $F_{\mu\nu}$.

After integrating out the bulk values of the string coordinates in the interior of the disc, the bosonic sector of the Polyakov path integral (2.1) at tree-level and in the conformal gauge becomes

$$Z_0[F] = \frac{1}{g_s}\int d\vec{x}_0 \int D\xi^\mu\ \mathrm{e}^{-S_\partial[\xi,A]}\,, \tag{2.5}$$

where the effective boundary action is

$$S_\partial[\xi,A] = \frac{1}{2}\int_0^{2\pi} d\vartheta\,\left(\frac{1}{2\pi\alpha'}\,\xi^\mu\,N^{-1}\,\xi_\mu + ieF_{\mu\nu}\,\xi^\mu\,\dot{\xi}^\nu\right). \tag{2.6}$$

Here N^{-1} denotes the coordinate space inverse of the boundary Neumann function (2.4) which is given explicitly by

$$N^{-1}(\vartheta,\vartheta') = -\frac{1}{\pi}\sum_{n=1}^\infty n\cos n(\vartheta-\vartheta')\,, \tag{2.7}$$

where we have used the completeness relation

$$\frac{1}{\pi}\sum_{n=1}^\infty \cos n(\vartheta-\vartheta') = \delta(\vartheta-\vartheta') - \frac{1}{2\pi} \tag{2.8}$$

for $\vartheta,\vartheta' \in [0,2\pi]$. The non-constant string modes ξ^μ can be written on $\partial\Sigma$ in terms of the Fourier series expansion for periodic functions on the circle,

$$\xi^\mu(\vartheta) = \sum_{n=1}^\infty\left(a_n^\mu\cos n\vartheta + b_n^\mu\sin n\vartheta\right). \tag{2.9}$$

The low-energy string effective action is then given by the renormalized value of (2.5).[2]

To evaluate the path integral (2.5), we use Lorentz invariance to rotate to a basis in which the antisymmetric $d\times d$ matrix $F_{\mu\nu}(x_0)$ is skew-diagonal with skew-eigenvalues f_ℓ, $\ell = 1,\ldots,d/2$. Then, on substituting (2.6), (2.7) and (2.9) into (2.5), the path integral factorizes into a product of $d/2$ functional gaussian integrations over the pairs of coordinate modes $a_n^{2\ell-1}, a_n^{2\ell}$ and $b_n^{2\ell-1}, b_n^{2\ell}$. The result of this integration yields a functional determinant and gives

$$Z_0[F] = \frac{1}{g_s}\int d\vec{x}_0\ \prod_{\ell=1}^{d/2} Z_{2\ell-1,2\ell}[f_\ell] \tag{2.10}$$

[2]It is a curious property of the Polyakov path integral that it computes directly the vacuum energy. The reason becomes clearer in the effective action approach [1] whereby conformal invariance is used to derive the variational equations of a spacetime effective action for the background fields. The string partition function is quite different from that of quantum field theory, in that it is more like an S-matrix.

where

$$Z_{2\ell-1,2\ell}[f_\ell] = \prod_{n=1}^{\infty}\left(\frac{4\pi^2\alpha'}{n}\right)^2\left[1+(2\pi\alpha' e f_\ell)^2\right]^{-1}. \tag{2.11}$$

The divergent infinite product $\prod_n \frac{1}{n^2}$ in (2.11) can be regulated using the ultraviolet cutoff ε and may be absorbed into a renormalization of the string coupling constant by using zeta-function regularization [25]. The other factor in (2.11) is also finite in zeta-function regularization

$$\prod_{n=1}^{\infty} c = c^{\zeta(0)}\,, \tag{2.12}$$

where $\zeta(s)=\sum_n \frac{1}{n^s}$ is the Riemann zeta-function with $\zeta(0)=-1/2$. We thereby find that $Z_{2\ell-1,2\ell}[f_\ell] = \frac{1}{4\pi^2\alpha'}\sqrt{1+(2\pi\alpha' e f_\ell)^2}$. By rotating back to general form, the regularized partition function (2.10) can be written in a Lorentz invariant way and it leads to the effective string action

$$S_{\rm BI} = \frac{1}{(4\pi^2\alpha')^{d/2} g_s}\int d\vec{x}_0\,\sqrt{\det_{1\le\mu,\nu\le d}\left[\delta_{\mu\nu}+2\pi\alpha' e F_{\mu\nu}\right]}\,, \tag{2.13}$$

which describes a model of non-linear electrodynamics for the field strengths that is governed by the classic Born-Infeld lagrangian [13].

The Euler-Lagrange equations for the action (2.13) can be written as

$$\sqrt{\det(\mathbb{1}+2\pi\alpha' eF)}\,\left(\frac{1}{\mathbb{1}-(2\pi\alpha' eF)^2}\right)_{\mu\nu}\beta_A^\nu = 0 \tag{2.14}$$

where

$$\begin{aligned}\beta_A^\mu &= \frac{\partial}{\partial\ln\varepsilon}\,\delta A^\mu(\varepsilon)\\ &= 2\pi\alpha' e\,\partial_\nu F^\mu_{\ \lambda}\left(\frac{1}{\mathbb{1}-(2\pi\alpha' eF)^2}\right)^{\lambda\nu}\end{aligned} \tag{2.15}$$

is the one-loop worldsheet β-function [1, 17], with $\delta A^\mu(\varepsilon)$ the cutoff dependent gauge field correction term that multiplies $\dot{x}^\mu$ in (2.2). The equations of motion for the gauge field are therefore equivalent to worldsheet conformal invariance of the quantum string theory. They are the stringy $\mathcal{O}(\alpha')$ corrections to the Maxwell field equations for A_μ. Notice that since $\det(\mathbb{1}+2\pi\alpha' eF)=\det(\mathbb{1}+2\pi\alpha' eF)^\top=\det(\mathbb{1}-2\pi\alpha' eF)$, the Born-Infeld action (2.13) contains only even powers of the field strength F in an α' expansion. The leading order term (the field theory limit) is given by the Maxwell lagrangian $-\frac{e^2T_s^{d/2-1}}{4(2\pi)^{d/2}g_s}F_{\mu\nu}^2$. For the uniform electromagnetic backgrounds that we shall deal with in most of this article, the calculations will thereby produce on-shell string amplitudes.

The Born-Infeld action is an example whereby the contributions in the coupling constant α', representing the string corrections to the field theory limit, can be summed to all orders of σ-model perturbation theory. Born-Infeld theory has many novel characteristics which distinguish it from the classical Maxwell theory of electromagnetism. These novel features are predominant for a purely electric background field, which in Minkowski space

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would have only non-vanishing temporal components $F_{0j} = iE_j$. Then, the electric field generated by a point-like charge is regular at the source and its total energy is finite [13]. The effective distribution of the field has a radius of the order of the string length scale $\sqrt{\alpha'}$, and the delta-function singularity is smeared away. This is quite unlike the situation in Maxwell theory, whereby the field of a point source is singular at the origin and its energy is infinite. The analogy with open string theory has been used to suggest that the terms of higher order in α' in the string effective action may eliminate Schwarzschild black hole singularities. Furthermore, at the origin of the source the electric field takes on its maximum value $|\vec{E}| = E_c$. The Born-Infeld lagrangian in this case takes the form $\sqrt{1-(2\pi\alpha' e\vec{E}\,)^2}$, which shows that there is a limiting value $E_c = T_s/e$ such that for $|\vec{E}| > E_c$ the action becomes complex-valued and ceases to make physical sense [25, 14, 45]. This instability reflects the fact that the electromagnetic coupling of strings is not minimal [5] and creates a divergence due to the fast rising density of string states. For field strengths larger than the critical electric field value E_c, the string tension T_s can no longer hold the strings together. We shall encounter other novel aspects of strings in background electric fields throughout this paper. Notice, however, that such novel effects and instabilities do not arise in purely magnetic backgrounds.

Going back to the case of euclidean signature, this calculation may be extended to the next order in string perturbation theory, whose contribution is the annulus diagram Σ which by scale invariance can be taken to have outer radius 1 and inner radius $a = \mathrm{e}^{-\pi t} \in [0,1]$. The variable a is therefore the modulus of the annulus and the path integration in (2.1) over metrics g_{ab} on Σ reduces, after gauge fixing, to an integral over Teichmüller space. We may now couple one endpoint of an open string to the boundary at $r = a$ with a charge e_1, and the other end at $r = 1$ with charge e_2, so that the one-loop action in the conformal gauge reads

$$S_1[x,A] = \frac{1}{4\pi\alpha'}\int d^2z\,\partial x^\mu\,\overline{\partial}x_\mu + ie_2\int_0^{2\pi} d\vartheta\,\dot{x}^\mu\,A_\mu(x)\Big|^{r=1} - ie_1\int_0^{2\pi} d\vartheta\,\dot{x}^\mu\,A_\mu(x)\Big|^{r=a}. \tag{2.16}$$

Here we shall consider only the case of neutral strings, $e_1 = e_2 = e$.[3] Charged strings will be dealt with later on.

Again, by using the method of images the Neumann function on the annulus is found to be given by the infinite series

$$\begin{aligned} N(z,z') = \frac{1}{2\pi}\Bigg[\ln|z-z'| &+ \sum_{n=1}^\infty \ln\left|1-\frac{a^{2n}}{z\overline{z}'}\right|\left|1-a^{2n-2}\,z\overline{z}'\right| + \\ &+ \sum_{n=1}^\infty \ln\left|1-\frac{a^{2n}\,z}{z'}\right|\left|1-\frac{a^{2n}\,z'}{z}\right|\Bigg], \end{aligned} \tag{2.17}$$

which satisfies the usual equation of motion and the Neumann boundary conditions $\partial_r N(z,z')|_{r=a} = 0$, $\partial_r N(z,z')|_{r=1} = \frac{1}{2\pi}$ [1]. At the worldsheet boundaries where $z_k = \mathrm{e}^{i\vartheta_k}$,

[3]Because of the orientation reversal between the two string endpoints, the net charge of an oriented open string is $e_1 - e_2$.

$k = 1, 2$, the annulus Green's function (2.17) can be written as

$$N(\vartheta_k, \vartheta_l') = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{[G_n]_{kl}}{n} \cos n \left(\vartheta_k - \vartheta_l'\right) , \tag{2.18}$$

where $k, l = 1, 2$ and G_n is the 2×2 matrix

$$G_n = \begin{pmatrix} A_n & B_n \\ B_n & A_n \end{pmatrix} \tag{2.19}$$

with

$$A_n = \frac{1 + a^{2n}}{1 - a^{2n}} , \qquad B_n = \frac{2a^n}{1 - a^{2n}} . \tag{2.20}$$

The function (2.18) is easy to invert and proceeding as before the one-loop effective action may thereby be calculated to be [25, 1]

$$Z_1[F] = Z_1[0] \int d\vec{x}_0 \det_{1 \le \mu, \nu \le d} \left[\delta_{\mu\nu} + 2\pi\alpha' e F_{\mu\nu} \right] , \tag{2.21}$$

where

$$Z_1[0] = \frac{g_s}{2} \int_0^{\infty} \frac{dt}{t} \left(4\pi^2 \alpha' t\right)^{-13} \eta \left(\frac{it}{2}\right)^{-24} \tag{2.22}$$

is the usual zero field vacuum energy for the annulus in the bosonic critical dimension $d = 26$, and $\eta(\tau)$ is the Dedekind function. The partition function (2.22) contains the contribution from the two conformal ghost fields which do not couple to the external field $F_{\mu\nu}$.

This result may be straightforwardly extended to fermionic strings by using the usual coupling of a spinor particle to an electromagnetic field and by using anti-periodic Fourier series expansions on $\partial\Sigma$ for the fermion fields and the corresponding string propagator. At tree level, the only effect of supersymmetry is to cancel the tachyonic divergence that arises in (2.11) [57]. The final result is again the Born-Infeld action (2.13). The extension to non-abelian gauge fields is also straightforward [55] and yields the effective non-abelian Born-Infeld action for open strings whose endpoints transform in the fundamental representation of the gauge group. The leading term in the α' expansion is the usual Yang-Mills lagrangian for the non-abelian gauge field. Demanding spacetime supersymmetry then leads to the usual low-energy effective field theory description in terms of maximally supersymmetric Yang-Mills theory in $d = 10$ spacetime dimensions (the superstring critical dimension).

2.2 Open string spectrum

In this subsection we will describe the spectrum of open strings in a constant background electromagnetic field in second quantization using the operator formalism [1]. We will concentrate again on bosonic strings, as we are merely interested here in some of the basic qualitative features of the spectrum. We will assume that the string worldsheet Σ is now an infinite strip with coordinates (τ, σ), where $\tau \in \mathbb{R}$ and $\sigma \in [0, 1]$ (This surface is conformally

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equivalent to the disc). The euclidean action is given by

$$
\begin{aligned}
S_{\rm strip} = {} & \frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\ \left(\partial_\tau x^\mu\,\partial_\tau x_\mu + \partial_\sigma x^\mu\,\partial_\sigma x_\mu\right) + \frac{e_2}{2}\int d\tau\ F_{\mu\nu}\,x^\nu\,\partial_\tau x^\mu\Big|^{\sigma=1} - \\
& - \frac{e_1}{2}\int d\tau\ F_{\mu\nu}\,x^\nu\,\partial_\tau x^\mu\Big|^{\sigma=0}\ ,
\end{aligned}
\tag{2.23}
$$

and in the worldsheet canonical formalism we regard τ as the time coordinate and σ as the space coordinate (so that Σ now has Minkowski signature). Varying (2.23) gives the usual wave equation $\Box\, x^\mu = 0$ along with the mixed Neumann-Dirichlet boundary conditions

$$
\begin{aligned}
\left(\partial_\sigma x^\mu - 2\pi\alpha' e_1 F^\mu_{~\nu}\,\partial_\tau x^\nu\right)_{\sigma=0} &= 0\,, \\
\left(\partial_\sigma x^\mu - 2\pi\alpha' e_2 F^\mu_{~\nu}\,\partial_\tau x^\nu\right)_{\sigma=1} &= 0\,.
\end{aligned}
\tag{2.24}
$$

We will again use Lorentz invariance to skew diagonalize the real-valued antisymmetric tensor $F_{\mu\nu}$. Since the skew blocks are independent, it suffices to concentrate on only one of them, and so we assume that the only non-vanishing component of the field strength tensor is $F_{01} = -F_{10} = F$. In this plane of the field, we introduce the complex target space coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm i x^1)$, in terms of which the boundary conditions (2.24) become

$$
\begin{aligned}
\left(\partial_\sigma x^+ + 2\pi i\alpha' e_1 F\,\partial_\tau x^+\right)_{\sigma=0} &= 0\,, \\
\left(\partial_\sigma x^- + 2\pi i\alpha' e_2 F\,\partial_\tau x^-\right)_{\sigma=1} &= 0\,,
\end{aligned}
\tag{2.25}
$$

along with the standard free open string Neumann boundary conditions in all of the directions transverse to the 0–1 plane.

We will now write down mode expansions which solve the equations of motion and satisfy the requisite boundary conditions (2.25). Since the only modification from the usual free string case occurs for the harmonic string coordinates in the 0–1 plane, we will focus our attention on their contributions. For this, it is necessary to treat neutral and charged open strings separately. As we will see in the following, there are drastic differences at both a qualitative and analytic level between the two cases. Let us note, however, that unitarity of the open string theory always requires the existence of both charged and neutral strings in the spectrum [3]. Consider a string scattering amplitude with the given charges e_1, e_2 at the endpoints. An amplitude with an even number of external legs can be sliced in many different ways into intermediate states. Some of these intermediate states will consist of open strings with either the charge e_1 or e_2 at both of its ends. An amplitude with an odd number of external legs necessarily involves at least one neutral string state in the scattering process. Therefore, any amplitude should be summed over *all* charges in the decomposition of the fundamental representation of the Chan-Paton gauge group under the embedding of U(1) induced by the background electromagnetic field.

2.2.1 Neutral strings

Let us begin with the case where the total charge of the open string vanishes, $e_1 = e_2 = e$. In this case there is the freedom to add to the coordinates $x^\pm$ terms proportional to

$\tau \mp 2\pi^2 i\alpha' e\sigma$, which satisfy the boundary conditions (2.25) when $e_1 - e_2 = 0$. The mode expansions in the 0–1 plane can thereby be written as

$$
\begin{aligned}
x^{\pm}(\tau,\sigma) =\ & \frac{y^{\pm} + q_{\mp}\left[\tau \mp 2\pi^2 i\alpha' eF(\sigma - \frac{1}{2})\right]}{\sqrt{1 + (2\pi\alpha' eF)^2}} + \\
& + i\sum_{n=1}^{\infty}\left(\frac{a_n^{\pm}}{n}\mathrm{e}^{-in\tau}\cos\left(n\pi\sigma \pm \arctan 2\pi\alpha' eF\right) - \right. \\
& \qquad \left. - \frac{(a_n^{\mp})^{\dagger}}{n}\mathrm{e}^{in\tau}\cos\left(n\pi\sigma \mp \arctan 2\pi\alpha' eF\right)\right),
\end{aligned}
\tag{2.26}
$$

with $(y^{\pm})^{\dagger} = y^{\mp}$ and $(q_{\pm})^{\dagger} = q_{\mp}$. The expansions (2.26) are defined in terms of an orthonormal system of oscillation modes [1, 14, 45] which solves the variational problem for the action (2.16) on the infinite strip and which diagonalizes it. The canonical momenta conjugate to the fields $x^{\pm}$ are given by $p_{\pm} = \partial_{\tau} x^{\pm}$ and they lead to the canonical commutation relations of the quantum string theory in the usual way. Because of the Born-Infeld factor in the denominator of the first line of (2.26), one finds that the zero mode positions $y^{\pm}$ and momenta $q_{\pm}$ obey canonical (Heisenberg) commutation relations, respectively, and are mutually commutative otherwise. The Fourier modes obey the Heisenberg-Weyl algebra $[a_m^{\pm}, (a_n^{\mp})^{\dagger}] = n\,\delta_{nm}$. They therefore satisfy the same harmonic oscillator commutation relations that they would in the absence of the external field.

The hamiltonian density is $(\partial_{\tau} x^{+} + \partial_{\sigma} x^{+})(\partial_{\tau} x^{-} + \partial_{\sigma} x^{-})$ which leads to the total worldsheet hamiltonian

$$
L_0^{\|} = q_+ q_- + \sum_{n=1}^{\infty}\left((a_n^{-})^{\dagger}\, a_n^{+} + (a_n^{+})^{\dagger}\, a_n^{-}\right)
\tag{2.27}
$$

in the 0–1 plane, with $q_+ q_- = \frac{1}{2}\,(q_1^2 + q_2^2)$. We conclude that the spectrum of a neutral open string is not affected by the electromagnetic field. However, as we saw in the previous subsection, the vacuum-to-vacuum amplitude is modified because the usual Born-Infeld factor appears in the mass-shell condition [1].

2.2.2 Charged strings

When the total charge of the string is non-vanishing, the entire structure of the external field problem is different. The string fields no longer have integer oscillator modes, and the zero modes change completely. In particular, there is no function linear in τ and σ which can satisfy the boundary conditions (2.25) when $e_1 - e_2 \neq 0$. The mode expansion in this case is given by

$$
\begin{aligned}
x^{\pm}(\tau,\sigma) =\ & y^{\pm} \mp i\, a_0^{\pm}\,\frac{\mathrm{e}^{\pm i\alpha\tau}}{\alpha}\cos\left(\pi\alpha\sigma - \arctan 2\pi\alpha' e_1 F\right) + \\
& + i\sum_{n=1}^{\infty}\left[\frac{a_n^{\pm}}{n \mp \alpha}\,\mathrm{e}^{-i(n\mp\alpha)\tau}\cos\Big((n \mp \alpha)\pi\sigma \pm \arctan 2\pi\alpha' e_1 F\Big) - \right. \\
& \qquad \left. - \frac{(a_n^{\mp})^{\dagger}}{n \pm \alpha}\mathrm{e}^{i(n\pm\alpha)\tau}\cos\Big((n \pm \alpha)\pi\sigma \mp \arctan 2\pi\alpha' e_1 F\Big)\right],
\end{aligned}
\tag{2.28}
$$

where

$$\alpha = \frac{1}{\pi}\Big(\arctan 2\pi\alpha' e_1 F - \arctan 2\pi\alpha' e_2 F\Big)\,. \tag{2.29}$$

We will assume in this subsection that $e_1 > e_2$ so that $\alpha > 0$. The normal mode functions in (2.28) again diagonalize the action (2.16), and solve the wave equation and the boundary conditions (2.25) [1, 14, 45]. Note that since their integrals are non-zero, the $y^\pm$ cannot be identified with the center of mass coordinates of the open strings, in contrast to the neutral case. Notice also the appearence of the extra modes $a_0^\pm$ compared to the $\alpha = 0$ case, which by reality are required to be hermitean operators. Canonical quantization now identifies the quantum commutators

$$\Big[a_n^\pm\,,\,(a_m^\mp)^\dagger\Big] = (n \pm \alpha)\,\delta_{nm}\,, \tag{2.30}$$

$$\big[y^+\,,\,y^-\big] = \frac{1}{2\alpha' F}\,\frac{1}{e_1 - e_2}\,. \tag{2.31}$$

The drastic change in the zero mode structure for charged strings is apparent in the commutation relation (2.31), which is ill-defined in the neutral string limit $e_1 = e_2$.

The total worldsheet hamiltonian in the 0–1 plane can be worked out to be

$$L_0^{||} = \sum_{n=1}^{\infty}(a_n^+)^\dagger\,a_n^- + \sum_{n=0}^{\infty}(a_n^-)^\dagger\,a_n^+ + \frac{1}{2}\,\alpha(1-\alpha)\,. \tag{2.32}$$

The normal ordering constant in (2.32) depends on the (arbitrary) choice of a_0^+ as an annihilation operator and is required to put the Virasoro algebra in standard form [1, 45]. We see therefore that the external electromagnetic field has a drastic effect on the spectrum of charged strings. It shifts the oscillation frequencies by amounts $\pm\alpha$, it modifies the commutation relations of the zero modes, and it changes the zero point energy. Furthermore, the open string momentum operators $q_\pm$ no longer appear in the mode expansions, while there are extra Fourier operators $a_0^\pm$ which create and annihilate quanta of frequency α. In fact, the contribution from the coordinates in the 0–1 plane is formally identical to that of a twisted unprojected sector of an orbifold string theory with twist angle α. This orbifold analogy provides a computationally convenient characterization of the external field problem.

Normally, one would take the quantum states to be eigenstates of definite momentum. However, when $e_1 \neq e_2$, it is instead the zero mode operators $y^\pm$ that commute with the hamiltonian $L_0^{||}$, and so we may take the states to be eigenstates of y^+, for example. Note that the operator y^- is, according to (2.31), a conjugate momentum operator for y^+. Since $L_0^{||}$ does not depend on $y^\pm$, there is an infinite degeneracy in the spectrum. In fact, the present physical situation is identical to that of a charged particle moving in the plane under the influence of a perpendicular uniform magnetic field. The states form equally spaced Landau levels of infinite degeneracy, with the energy difference between consecutive levels proportional to $(e_1 - e_2)F$. The operators $a_0^\pm$ move the string from one Landau level to another, and their frequency separation (2.29) is proportional to the quantity $(e_1 - e_2)F$ when it is sufficiently small, i.e. in the weak-field limit $\alpha = 2\alpha'(e_1 - e_2)F + \mathcal{O}(F^3) \ll 1$. Deviations from the field theoretic result at strong fields F come from the non-minimality of the electromagnetic string coupling and are parametrized by the non-linear function α of the field. Excited states of the open string are then obtained by acting on the ground state wavefunctions with oscillator creation operators. At the first excited level, there are the states $(a_1^+)^\dagger|y^+\rangle$ with tachyonic mass $\sqrt{-\frac{1}{2}\,\alpha(1+\alpha)}$, and the states $(a_1^-)^\dagger|y^+\rangle$ with mass $\sqrt{\frac{1}{2}\,\alpha(3+\alpha)}$. This is reminescent of the situation that occurs in Yang-Mills theory in the presence of a chromomagnetic field condensate, whereby one gluonic polarization becomes unstable due to its tachyonic energy and the other becomes massive [46].[4] In fact, as we shall see in the following, the charged string system possesses many instabilities. Only the neutral open string makes sense both physically and analytically.

2.3 The Schwinger mechanism

In this subsection we will compute the one-loop vacuum energy for charged strings using the operator formalism of the previous subsection and elucidate somewhat on the instabilities that we have thus far encountered. In particular, we will examine the instability of the string vacuum in a purely electric background [14, 8]. This can be obtained from the calculations of the previous subsection by the analytical continuations $F=iE$ and $\alpha=-i\epsilon$ corresponding to Wick rotations of both the worldsheet and target space time coordinate to Minkowski signature. The vacuum energy may be computed as the logarithm of the partition function $\det(L_0-1)^{-1/2}$ of the underlying free conformal field theory σ-model, where $L_0=L_0^\perp+L_0^{||}+L_0^{\rm ghosts}$ is the total hamiltonian comprised of the contributions from the fields transverse to the plane of the electric field, those parallel to the 0–1 plane, and the conformal ghost fields. The annulus amplitude is thereby given as

$$-i\,V_d\,\mathcal{F}(e_1,e_2)=\frac{1}{2}\,{\rm tr}_{(e_1,e_2)}\ln\left(L_0-1\right)\,, \tag{2.33}$$

where V_d is the volume of spacetime and ${\rm tr}_{(e_1,e_2)}$ denotes the trace over all string states in the (e_1,e_2) charge sector. The total annulus amplitude is a sum over all allowed endpoint charges.

The trace (2.33) is straightforward to evaluate by using the proper time representation

$$\ln\mathcal{A}=-\int_0^\infty\frac{dt}{t}\,{\rm e}^{\,-\pi t\mathcal{A}}\,, \tag{2.34}$$

and the fact that for any set of oscillator operators a_n obeying Heisenberg-Weyl commutation relations there is the formula

$$\begin{aligned}{\rm tr}~{\rm e}^{\,-\pi t\sum_{n\geq1}a_n^\dagger\,a_n}&=\prod_{n=1}^\infty{\rm tr}~{\rm e}^{\,-\pi t\,a_n^\dagger\,a_n}\\&=\prod_{n=1}^\infty\,\sum_{m=0}^\infty{\rm e}^{\,-\pi tmn}=\prod_{n=1}^\infty\left(1-{\rm e}^{\,-\pi tn}\right)^{-1}\,,\end{aligned} \tag{2.35}$$

where we have used a basis of all possible multi-particle states. For the transverse degrees of freedom the oscillator traces are accompanied by $d-2$ gaussian momentum integrals, coming from the analogs of the first term in (2.27), and an integration over the canonically

[4] In contrast to the usual bosonic string tachyon state, this instability is not removed by supersymmetry [24].

conjugate zero modes $y^\perp$ of the string fields which produces a volume factor V_{d-2}. These are also multiplied by a factor $(2\pi)^{d-2}$ which is the density of quantum phase space states. For the fields along the 0–1 plane, we can use the identity (2.35) with the appropriate shifts of oscillation frequencies given in (2.30). There is also an integration over the zero modes $y^\pm$ which contribute, according to (2.31), a quantum state density factor $\frac{\alpha'}{\pi}\,E\,(e_1-e_2)$, along with the volume of the 0–1 plane.

By incorporating the ghost contributions and putting all of these results together we arrive finally at

$$\mathcal{F}(e_1,e_2)=\frac{1}{2}\int_0^\infty\frac{dt}{t}\,\left(4\pi^2\alpha' t\right)^{-13}\,\eta\left(\frac{it}{2}\right)^{-24}\,C_A(t,E) \tag{2.36}$$

in the critical dimension $d=26$, where

$$C_A(t,E)=\alpha'(e_1-e_2)E\,t\,\mathrm{e}^{\,-\pi t\epsilon^2/2}\,\frac{\Theta_1'\left(0\,\middle|\,\frac{it}{2}\right)}{\Theta_1\left(\frac{\epsilon t}{2}\,\middle|\,\frac{it}{2}\right)} \tag{2.37}$$

is a field dependent correction factor. Here $\Theta_a(\nu|\tau)$ denote the standard Jacobi-Erderlyi theta-functions and $\Theta_1'(\nu|\tau)=\frac{\partial}{\partial\nu}\Theta_1(\nu|\tau)$. In the zero field limit the amplitude (2.36) reduces to the expected result (2.22), since $C_A(t,0)=1$. It gives the modification of the neutral string effective action (2.21) to the charged case. In the limit $e_1-e_2=\delta\to 0$, we have $\pi\epsilon=\frac{\delta}{1-(2\pi\alpha' e_1E)^2}+\mathcal{O}(\delta^2)$, and the correction factor (2.37) takes the simple t-independent form $C_A(t,E)=1-(2\pi\alpha' e_1E)^2+\mathcal{O}(\delta)$. Thus, for neutral strings the annulus amplitude (2.36) is proportional to the square of the Born-Infeld lagrangian for this case, as in (2.21).

The most interesting feature of the vacuum energy (2.36) is that it is imaginary. The theta-function appearing in the denominator of (2.37) contains a trigonometric function, $\Theta_1(\frac{\epsilon t}{2}|\frac{it}{2})\propto\sin(\frac{\pi t\epsilon}{2})$, and so the function $C_A(t,E)$ has simple poles on the positive t-axis at $t=2k/|\epsilon|$, $k=1,2,\dots$ The amplitude thereby acquires an imaginary part given by the sum of the residues at the poles times a factor of π, since, as dictated by the proper definition of the Feynman propagator, the contour of integration in the complex t-plane should pass to the right of all poles. What this quantum instability represents is the spontaneous creation of charged strings from the vacuum [14, 8], in analogy to the instability of the vacuum state in quantum electrodynamics [50]. By computing the corresponding residues of the function (2.37), the total rate of pair production is found to be given by

$$\begin{aligned} w_{\rm str} &= -2\,\mathrm{Im}\,\mathcal{F}\\ &= \frac{1}{(2\pi)^d}\sum_{e_1,e_2}\sum_S\frac{\alpha'(e_1-e_2)}{\epsilon}\sum_{k=1}^\infty(-1)^{k+1}\left(\frac{|\epsilon|}{k}\right)^{d/2}\,\mathrm{e}^{\,-\frac{2\pi k}{|\epsilon|}\left(M_S(\epsilon)^2+\frac{\epsilon^2}{2}\right)}\,, \end{aligned} \tag{2.38}$$

where the second sum runs through all physical open string states of mass $M_S(\epsilon)$ which may be computed from the generating function

$$\frac{\mathrm{e}^{\,\left(1-\frac{\epsilon^2}{2}\right)l}}{\sin\left(\frac{\epsilon l}{2}\right)}\,\prod_{n=1}^\infty\frac{\left(1-\mathrm{e}^{\,-nl}\right)^{-(d-4)}}{\left(1-\mathrm{e}^{\,-(n+i\epsilon)l}\right)\left(1-\mathrm{e}^{\,-(n-i\epsilon)l}\right)}=\sum_S\mathrm{e}^{\,-M_S(\epsilon)^2\,l}\,. \tag{2.39}$$

Note that neutral string states do not contribute to the pair production rate, as expected, and indeed the neutral string vacuum energy (2.21), (2.22) is real-valued.

The expression (2.38) represents the stringy modification of the classic Schwinger probability amplitude for pair creation of charged particles in a uniform external electric field E [50]. In that case the probability per unit volume and unit time is given by

$$w_0 = \frac{2J+1}{(2\pi)^3} \sum_{k=1}^{\infty} (-1)^{(2J+1)(k+1)} \left(\frac{QE}{k}\right)^2 \, \mathrm{e}^{\,-2\pi k M^2/|QE|} \,, \tag{2.40}$$

where Q, J and M are the charge, spin and mass of the created particles. In this quantum field theory calculation the imaginary part of the vacuum energy comes from the determinant $\det(-D_A^2)^{-1/2}$ of the massive gauge-covariant Dirac operator D_A, which at tree-level would produce the result $\frac{\pi M^{-1} QE}{\sin(\pi M^{-1} QE)}$. In fact, the result (2.38) coincides with (2.40) with $Q = 2\alpha'(e_1 - e_2)$ in the weak-field limit in $d = 4$ dimensions, since a particle-antiparticle pair of spin J has $2(2J+1)$ physical states.

However, in contrast to the field theory case, the string theory deteriorates at strong external fields. Since $\epsilon \to \infty$ as $E \to E_c = T_s/e_1$, the total rate for pair production diverges at the critical electric field [8]. Thus the classical instability of the string vacuum state in an electric field can also be seen at the quantum level. At this critical value of the external field, the string tension can no longer stop charged strings from nucleating out of the vacuum. In fact, this limiting instability also occurs for neutral strings. If we concentrate on only the first line of (2.26) (the zero mode contributions), then we see that the open string can be thought of as a rod, of length proportional to $E\,q$, which behaves like an electric dipole whose ends carry equal and opposite charges. When an open string is stretched along the direction of the background electric field, the field reduces its energy, and at $E = E_c$ the energy stored in the tension of the string is balanced by the electric energy of the stretched string. For $E > E_c$, virtual strings can materialize out of the vacuum, stretch to infinity and destabilize the ground state. In fact, from the first line of (2.26) we see that for fixed worldsheet time τ the two endpoints of the string are not at the same value of x^+, but they are always spacelike separated. As the electric field becomes critical the two ends at fixed τ become lightlike separated.

Of course, in the genuine type-I theory that these considerations really pertain to, one should add to the annulus amplitude the contribution from its non-orientable counterpart, the Möbius strip diagram. This is straightforward to do and the role of the Möbius amplitude is to project out the reflection-odd, neutral oriented string states [8].[5] One should also, for unitarity reasons, consider the contributions from the one-loop closed string diagrams, i.e. the torus and the Klein bottle. Since closed string states do not couple to the external field, their amplitudes are the same as in the zero field limit. The four contributions to the total vacuum energy now have an elegant interpretation in terms of the worldsheet orbifold construction that we mentioned earlier, whereby the torus and annulus diagrams give the contributions of untwisted and twisted sectors, respectively, while the addition of the Klein bottle and Möbius strip diagrams takes care of the projections onto states

[5]Since the Möbius strip has only a single connected boundary, only neutral string states contribute to the Möbius amplitude. Physical string states must also be even under a worldsheet parity reflection of the open strings.

which are even under the action of the orbifold group. The extension of the calculation to open superstrings is also straightforward and again one easily recovers the Schwinger formula (2.40) in the weak-field limit [8, 56, 57].

3. Thermal ensembles

In this section we will describe some properties of strings in background electromagnetic fields at finite temperature. For this, we are interested in computing the thermodynamic partition function

$$\mathcal{Z} = \mathrm{tr}\, \mathrm{e}^{\,-\beta(L_0-1)}\,, \tag{3.1}$$

where $\beta = 1/k_{\rm B}T$ with $k_{\rm B}$ the Boltzmann constant and T the temperature. Temperature represents another explicit supersymmetry breaking mechanism and it leads to a variety of novel effects in string theory. At the forefront of these exotic features is the influence of the density of single particle states on the thermodynamic properties of the string gas. The number of states at level N grows exponentially as $\mathrm{e}^{\,4\pi\sqrt{N}}$, which is so rapid that the thermodynamic partition function (3.1) of the free string gas converges only for sufficiently small temperatures $T < T_{\rm H}$, where

$$T_{\rm H} = \frac{1}{2\pi k_{\rm B}\sqrt{2\alpha'}} \tag{3.2}$$

is known as the Hagedorn temperature [31]. Generally, models with an exponentially rising density of states exhibit non-extensive thermodynamic quantities and a pair of such systems can never attain thermal equilibrium. However, in string theory the Hagedorn temperature is not a limiting temperature, because it requires a finite amount of energy to reach it in the canonical ensemble. Rather, it is associated with a phase transition, analogous to the deconfinement transition that occurs in Yang-Mills theory. The Hagedorn temperature $T_{\rm H}$ is the critical point at which infrared divergences emerge due to a closed string state becoming massless [34, 48, 6]. The Hagedorn transition may therefore be associated with the appearence of tachyonic winding modes. In the following we will examine how this picture is affected by the presence of background fields.

Although the thermodynamic ensemble of superstrings is interesting in its own right [39], the inclusion of electromagnetic fields will allow us in the next section to map the free string gas onto a system of D-branes. Thermal states of superstrings in electromagnetic fields thereby correspond to non-extremal states of D-branes in supergravity which have a natural Hawking radiation and entropy. They are therefore relevant to the microscopic description of black holes in string theory. Since the string theory universally contains gravity, the system destabilizes at finite energy density in the thermodynamic limit. This is due to the Jeans instability which occurs because a relativistic thermal ensemble at sufficiently large volume reaches its Schwarzschild radius and collapses into a black hole [6].

In the path integral approach to finite temperature string theory, the spacetime is taken as euclidean space with time x^0 compactified on a circle of circumference β. Temperature affects the string gas because the string can wrap around the compact time direction with a given winding number $n \in \mathbb{Z}$, i.e. $x^0(r,\vartheta+2\pi) = x^0(r,\vartheta) + n\beta$. This affects only the zero modes of the bosonic string embedding field x^0 and can be incorporated by adding a term $\frac{n\beta}{2\pi}\,\vartheta$ to its mode expansion. In string perturbation theory, the disc amplitude (2.13) is unmodified at finite temperature, because the disc worldsheet cannot wrap the cylindrical target space and so cannot distinguish between a compactified and an uncompactified spacetime. The first corrections due to temperature appear in the annulus amplitude. We can evaluate the thermodynamic free energy $\mathcal{F} = -\ln(\mathcal{Z})/\beta$ as before by computing the Polyakov path integral for the action (2.16) and enforcing the periodicity constraint in euclidean time via the substitution

$$x^0(r,\vartheta) \mapsto x^0(r,\vartheta) + \frac{n\beta}{2\pi}\,\vartheta\,. \tag{3.3}$$

We then sum the path integral over all temperature winding modes $n \in \mathbb{Z}$.

For a uniform electromagnetic field, there are two commonly used gauge choices, the static gauge $(A_0, A_i) = (-\vec{F}\cdot\vec{x}, \frac{1}{2}\,F_{ij}\,x^j)$ and the temporal gauge $(A_0, A_i) = (0, F_i\,x^0 + \frac{1}{2}\,F_{ij}\,x^j)$. Here the vector $\vec{F}$ denotes the temporal components of the euclidean field strength tensor, $F_i = F_{0i}$, $i = 1,\dots,d-1$, which is related to the electric field $\vec{E}$ in Minkowski space by $\vec{F} = i\vec{E}$. In the temporal gauge, the gauge potential is only periodic in euclidean time up to a gauge transformation, and in this case it is necessary to augment the usual coupling of the edge of a charged open string to the gauge field by adding a generalized Wu-Yang term [3] in order to compensate the gauge transformation. Here we shall choose the periodic static gauge field configuration. On substituting (3.3) into (2.16) the action then reads

$$\begin{aligned} S_1[x,A] = {} & \frac{1}{4\pi\alpha'}\int d^2z\ \partial x^\mu\,\overline{\partial} x_\mu + \frac{n^2\beta^2 t}{32\pi\alpha'} + \\ & + \frac{ie_2 n\beta}{2\pi}\int_0^{2\pi} d\vartheta\ \vec{F}\cdot\vec{x}\,\Big|^{r=1} - \frac{ie_1 n\beta}{2\pi}\int_0^{2\pi} d\vartheta\ \vec{F}\cdot\vec{x}\,\Big|^{r=a} + \\ & + ie_2\int_0^{2\pi} d\vartheta\ \dot{x}^0\,\vec{F}\cdot\vec{x}\,\Big|^{r=1} - ie_1\int_0^{2\pi} d\vartheta\ \dot{x}^0\,\vec{F}\cdot\vec{x}\,\Big|^{r=a} + \\ & + \frac{ie_2}{2}\int_0^{2\pi} d\vartheta\ \dot{x}^i\,F_{ij}\,x^j\,\Big|^{r=1} - \frac{ie_1}{2}\int_0^{2\pi} d\vartheta\ \dot{x}^i\,F_{ij}\,x^j\,\Big|^{r=a}\,. \end{aligned} \tag{3.4}$$

It is clear from (3.4) that electric and magnetic backgrounds contribute very differently at finite temperature, and so we shall analyse their couplings separately.

3.1 Magnetic fields

Let us begin with the purely magnetic case and set $\vec{F} = \vec{0}$ in (3.4). Let f_ℓ, $\ell = 1,\dots,\frac{d-2}{2}$, be the skew-eigenvalues of the magnetic flux tensor F_{ij}. The zero temperature annulus amplitude when F_{ij} has only a single non-vanishing skew-eigenvalue f_ℓ is given by (2.36), (2.37) with $E = -if_\ell$ and $\epsilon = i\alpha(f_\ell)$, and it is real-valued. Since the mode expansions of the string fields are the same as before, it follows that the only effect of finite temperature comes from the constant term in the first line of (3.4). By summing the path integral over all thermal winding modes, this inserts into the Teichmüller integration defining the annulus amplitude the infinite series

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\,-\beta^2 n^2 t/32\pi\alpha'} = \Theta_3\left(0\left|\frac{i\beta^2 t}{32\pi^2\alpha'}\right.\right). \tag{3.5}$$

The one-loop free energy per unit volume of the string gas in the magnetic background is thereby given as

$$\mathcal{F}_{\rm mag} = \frac{1}{2}\int_0^\infty \frac{dt}{t}\left(4\pi^2\alpha' t\right)^{-13}\eta\left(\frac{it}{2}\right)^{-24}\Theta_3\left(0\left|\frac{i\beta^2 t}{32\pi^2\alpha'}\right.\right)\prod_{\ell=1}^{12} C_A(t,-if_\ell)\,. \tag{3.6}$$

We will now examine the convergence properties of the Teichmüller integral (3.6).

The open string ultraviolet behaviour is determined by the $t \to 0$ region of moduli space. In this limit the theta functions appearing in (3.6), (2.37) have the asymptotics $\eta(\frac{it}{2})^{-24} \sim t^{12}\,\mathrm{e}^{\,4\pi/t}$, $\Theta_1(\frac{i\alpha t}{2}|\frac{it}{2}) \sim t^{-1/2}\,\mathrm{e}^{\,-\pi/2t}\,\mathrm{e}^{\,\pi\alpha^2 t/2}\sin\pi\alpha$, and $\Theta_3(0|\frac{i\beta^2 t}{32\pi^2\alpha'}) - 1 \sim t^{-1/2}\,\mathrm{e}^{\,-32\pi^3\alpha'/\beta^2 t}$. From these behaviours it follows that the integral (3.6) converges in the region $t \to 0$ provided that $\beta > 1/k_{\rm B}T_{\rm H}$, where $T_{\rm H}$ is the Hagedorn temperature (3.2).[6] Note that the overall asymptotics are completely independent of the external field, and we therefore conclude that the presence of a magnetic field does not change the value of the Hagedorn temperature of the free open string gas. The open string infrared behaviour, on the other hand, comes from the $t \to \infty$ region of moduli space. In this limit the temperature dependence of (3.6) disappears, $\Theta_3(0|\frac{i\beta^2 t}{32\pi^2\alpha'}) \sim 1$, and the conditions for convergence of the integral are the same as at zero temperature. One encounters the infrared magnetic instability that was described in section 2.2.2 [56]. This instability in the thermodynamic free energy is also present in supersymmetric Yang-Mills theory at both zero and finite temperature.

It is straightforward to repeat the analysis for the fermionic string (where there is no zero temperature tachyon mode). As spacetime bosons are required to obey periodic boundary conditions along the euclidean time circle and spacetime fermions to obey anti-periodic ones, supersymmetry is explicitly broken and the GSO projection must be accordingly modified. At the one-loop level this may be achieved by inserting an extra weighting $(-1)^n$ in the sum over temperature winding numbers for the $(-,+)$ spin structure in the Neveu-Schwarz sector of the worldsheet theory [6]. Then one can compute that the instabilities encountered above persist for superstrings [56].

3.2 Electric fields

Now let us consider the case of a purely electric background and set $F_{ij} = 0$ in (3.4). From the second line of (3.4) we see that there is now a non-trivial coupling between the electric field and the temperature winding modes. This coupling has several dramatic effects on the thermal ensemble. The most glaring one is that it prevents the formation of an equilibrium distribution of charged strings in the electric field [3]. To see this, we

[6] Of course, the bosonic string theory has a tachyonic instability at any temperature. The Hagedorn temperature is defined in this case as the temperature at which the one-loop free energy of the bosonic string gas diverges even if the contribution from the usual tachyon mode is ignored.

consider the zero modes $x^\mu = x_0^\mu$ of the string embedding fields on the annulus. In the absence of an electric field, the action is independent of them and integrating them out in the path integral produces a volume factor βV_{d-1}. In the present case, however, they contribute the quantity

$$\beta \int d\vec{x}_0 \ \mathrm{e}^{\,i(e_2-e_1)n\beta\vec{F}\cdot\vec{x}_0} = \beta\,(2\pi)^{d-1}\,\delta\Big((e_2-e_1)n\beta\vec{F}\Big)\,. \tag{3.7}$$

This result shows that thermal states of the string are stable only either for neutral strings or in the absence of the external field. All states except the ground state $n = 0$ contain excitations of charged particles and therefore have infinite energy. In fact, because of the Schwinger mechanism that we described in section 2.2, even the ground state is unstable. The breaking of translational invariance forbids an equilibrium state of charged strings in a constant background electric field. In what follows we shall describe some origins of this instability.

3.2.1 Neutral strings

It is natural to consider the neutral string case, $e_1 = e_2 = e$. In that case the zero modes $\vec{x}_0$ disappear from the action (3.4), but the second line still contributes a linear term to the gaussian form for the oscillatory modes $\vec{\xi}(\vartheta)$ of the string fields. On completing the square, this adds an extra term $\beta^2n^2t\alpha'e^2E^2/8$ to the argument of the exponential in the infinite series (3.5), where $E = -i|\vec{F}\,|$. The modification of the annulus amplitude (2.21), (2.22) to finite temperature is therefore the free energy

$$\begin{aligned}\mathcal{F}_{\rm el} = \frac{1}{2}\left(1-(2\pi\alpha' eE)^2\right)\int_0^\infty \frac{dt}{t}\left(4\pi^2\alpha' t\right)^{-13}\times \\ \times\,\eta\left(\frac{it}{2}\right)^{-24}\ \Theta_3\left(0\left|\frac{i\beta^2 t\left[1-(2\pi\alpha' eE)^2\right]}{32\pi^2\alpha'}\right.\right)\,.\end{aligned} \tag{3.8}$$

By repeating the asymptotic analysis of the previous subsection, we find that the Teichmüller integral (3.8) converges in the open string ultraviolet regime provided that $\beta > 1/k_{\rm B}T_{\rm H}(E)$, where

$$T_{\rm H}(E) = \frac{\sqrt{1-(2\pi\alpha' eE)^2}}{2\pi k_{\rm B}\sqrt{2\alpha'}}\,. \tag{3.9}$$

Thus, in contrast to the magnetic case, an electric background modifies the Hagedorn temperature (3.2) of the neutral string gas by the familiar Born-Infeld lagrangian [23].

This result was to be expected because, unlike magnetic fields, electric fields couple to the temporal coordinate and therefore scale the momentum of the strings. In turn, they rescale the proper time variable t. Note that the Hagedorn temperature (3.9) decreases with increasing electric field and vanishes at the critical value $E = E_c$. As is apparent in second quantization [45, 23] (see eq. (2.26)), the field dependent rescaling factor originates as a modification of the string tension $T_s = 1/2\pi\alpha'$, which determines the critical temperature (3.2), to an effective tension

$$T_{\rm eff} = T_s\left[1-(T_s^{-1}eE)^2\right] \tag{3.10}$$

which vanishes at the critical electric field. This modification of the tension is the reason why the thermodynamic properties of the neutral string gas are altered by the electric background. Again the same conclusions are reached for the full superstring free energy [56].

3.2.2 Charged strings

Restricting the spectrum to neutral open strings does not completely cure the electric field instability, because we have to sum over all allowed neutral and charged string states. We will now examine the reasons why finite temperature string theory forbids constant electric fields. This instability can in fact be seen at the field theoretical level. The coordinate space diagonal elements of the (un-normalized) thermal density matrix in quantum electrodynamics for a charged (scalar) particle of mass M and charge Q in a uniform electric field $\vec{E}$ is given by the proper time integral [40, 3]

$$\rho(\vec{y},\vec{y};\beta) = \frac{\beta}{2}\int_0^\infty \frac{d\tau}{\tau}\,\frac{\mathrm{e}^{\,-M^2\tau/2}}{(2\pi\tau)^{\frac{d}{2}-1}}\,\frac{QE}{4\pi\sin\frac{QE\tau}{2}}\,\Theta_3\left(\frac{i\beta Q\vec{E}\cdot\vec{y}}{2\pi}\,\middle|\,\frac{i\beta^2 QE}{4\pi\tan\frac{QE\tau}{2}}\right). \tag{3.11}$$

This result shows that the free energy of the system is trivial, in that the integration of (3.11) over $\vec{y}$ picks up only the ground state of winding number $n = 0$ (due to the occurence of the theta-function Θ_3). Note that the density matrix (3.11) is complex-valued and its imaginary part gives the Schwinger probability amplitude (2.40). Naively, the translational symmetry which is broken by the external electric field could be restored by choosing the temporal gauge for the gauge potential rather than the static gauge. However, this gauge choice ruins the global gauge invariance of the system. The gauge potential in this case is not a function on spacetime because it is multi-valued under periodic shifts around the temperature circle, and it can only be properly defined with respect to a local covering of the thermal direction. Requiring that the theory be independent of the choice of covering requires the addition of generalized Wu-Yang terms to the action (The mathematical details of this construction can be found in [3]). These terms restore gauge invariance, but they also reinstate precisely the same $\vec{y}$-dependent factor in (3.11). Therefore, the gauge-invariant free energy remains trivial.

For a thermal state of charged strings, the constraint (3.7) forces us to take $\vec{F} = \vec{0}$. This selects the constant gauge field configuration $A_0(x) = a_0$. Although this field is pure gauge, it can only be removed by a singular gauge transformation. Therefore, charged states will still depend on a_0, or equivalently the canonical, gauge invariant momentum of the open strings depends on a_0. The gauge field background cannot be removed because there is a non-trivial gauge invariant holonomy $\mathrm{e}^{\,i(e_2-e_1)na_0}$ which arises from the boundary terms in the action (2.16) in the sector of temperature winding number n. This holonomy is simply the Polyakov loop operator for the annular geometry. We can therefore study the free energy for this constant gauge field configuration and compute the effective action for charged strings in a generic, time-independent background gauge field $a_0(\vec{x}_0)$. This yields the free energy that is required to introduce a heavy charged particle into the system and thereby gives information about confinement, which is the pertinent property of the Hagedorn transition.

The action is now given by adding to the first line of (3.4) the term $i(e_2-e_1)na_0$. After summing over all $n\in\mathbb{Z}$, the appropriate modification of the one-loop vacuum energy (2.22) is given by

$$\mathcal{F}_0 = \frac{1}{(8\pi^2\alpha')^{13}}\int_0^\infty ds\eta(2is)^{-24}\Theta_3\left(\frac{(e_2-e_1)\beta}{2\pi}\,a_0\,\bigg|\,\frac{i\beta^2}{32\pi^2\alpha' s}\right), \tag{3.12}$$

where we have made a modular transformation $t=1/s$ (mapping the one-loop open string annulus diagram onto the tree-level closed string cylinder diagram) and used the Poisson resummation formula $\eta(-\frac{1}{\tau})=\sqrt{-i\tau}\,\eta(\tau)$. The integral (3.12) can be evaluated explicitly by expanding the Dedekind function using the formula

$$\prod_{n=1}^\infty\left(1-\mathrm{e}^{\,-nl}\right)^{-(d-2)} = \sum_{N=0}^\infty d_N^{\rm b}\,\mathrm{e}^{\,-Nl}\,, \tag{3.13}$$

where $d_N^{\rm b}$ is the degeneracy of bosonic string states at level N. For the first two levels we have $d_0^{\rm b}=1$ and $d_1^{\rm b}=d-2$. By expanding the theta-function Θ_3 in an infinite series we can thereby perform the integral (3.12) and arrive at

$$\mathcal{F}_0 = \frac{2\beta}{(8\pi^2\alpha')^{14}}\sum_{N=0}^\infty\frac{d_N^{\rm b}}{\sqrt{N}}\sum_{n=1}^\infty n\,K_1\left(n\beta\sqrt{\frac{N}{2\alpha'}}\right)\cos n\beta(e_1-e_2)a_0\,, \tag{3.14}$$

where $K_1(z)$ is the irregular modified Bessel function of order 1. The $N=0$ contribution to (3.14) of course diverges because of the tachyonic instability. The next contribution is from the level $N=1$, corresponding to the 26-dimensional Yang-Mills multiplet, which is well-defined. From the asymptotic expansion $K_1(z)\sim\mathrm{e}^{\,-z}\sqrt{\pi/2z}$ for $|z|\to\infty$, it follows that its contribution is suppressed in the low-temperature limit $\beta\gg\sqrt{\alpha'}$ by terms of order $\mathrm{e}^{\,-\beta/\sqrt{\alpha'}}$.

Nevertheless, this calculation illustrates the general feature whose instabilities are cured by computing the one-loop free energy of the superstring gas [3]. Then the lowest $N=0$ level yields a finite contribution in the low temperature limit which corresponds to the ten dimensional Yang-Mills supermultiplet. By including the tree-level Born-Infeld actions for the disc amplitudes of the charged string endpoints, we arrive at the total (normalized) effective action for the gauge field $a_0(\vec{x}_0)$ up to one-loop order in the form

$$\begin{aligned}\Gamma[a_0] &= \frac{1}{(2\pi\alpha')^2 g_s\left(e_1^2+e_2^2\right)}\\ &\quad\times\int d\vec{x}_0\Bigg[\sum_{k=1,2}\sqrt{1+\left(2\pi\alpha' e_k\nabla a_0(\vec{x}_0)\right)^2}+\\ &\qquad\qquad+\frac{\mu^2}{\beta^2(e_1-e_2)^2}\cos\beta(e_1-e_2)a_0(\vec{x}_0)+\mathcal{O}\left(g_s^4\,,\,\mathrm{e}^{\,-\beta/\sqrt{\alpha'}}\right)\Bigg],\end{aligned} \tag{3.15}$$

where

$$\mu^2 = 3\pi^2\cdot 2^{22}g_s^2(\alpha')^3\beta^{-8}\left(e_1^2+e_2^2\right)(e_1-e_2)^2\,. \tag{3.16}$$

Thus the low temperature modification of the Born-Infeld action is a generalization of the sine-Gordon theory representation of the classical Coulomb gas where the standard kinetic term for the gauge field is replaced by the Born-Infeld lagrangian. The main feature of this field theory is that the linearized equation of motion for the minima of the free energy (3.15) takes the form $(-\nabla^2 + \mu^2)a_0(\vec{x}_0) = 0$, which has exponentially decaying solutions $a_0(\vec{x}_0) \sim \mathrm{e}^{-\mu|\vec{x}_0|}$ for $|\vec{x}_0| \to \infty$. In this approximation the constant (3.16) appears as a mass term for the gauge field in (3.15) and acts like a Debye screening mass. It is clear for this reason that constant electric fields cannot be extrema of the effective action, i.e. the existence of uniform electric fields is inconsistent with the existence of a Debye mass. Note that (3.16) vanishes for neutral string states.

As expected, the Debye mass (3.16) is the same as the one that would arise in ordinary ten dimensional Yang-Mills theory. The contributions from massive string states are exponentially suppressed by terms of order $\mathrm{e}^{-\beta/\sqrt{\alpha'}}$. Since $\mu^2 \propto T/T_{\mathrm{H}}(0)$, for temperatures well below the critical Hagedorn temperature the Debye mass is small and electric fields become more and more long-ranged. Stringy effects essentially only play a role at temperatures near the Hagedorn transition. Furthermore, the Born-Infeld generalization (3.15) of the sine-Gordon model has solitons which generalize the solitary waves of the plasma phase of the ordinary Coulomb gas [3]. In gauge field theories these solitons exist as $\mathbb{Z}_N$ domain walls. This characterization could prove useful for other aspects of the Hagedorn transition in background fields.

4. D-brane dynamics

T-duality maps free open strings to open strings whose endpoints are attached to D-branes. It replaces the quantities $\partial_a x^i$ by $i\epsilon_{ab}\partial_b x^i$ and Neumann boundary conditions for the string embedding fields x^i with Dirichlet ones. The results of the previous sections may be interpreted directly as the appropriate contributions to the tension of a D9-brane with some background field distribution. T-dualizing these expressions in $9-p$ of the spacetime directions is then tantamount in string perturbation theory to adding an extra open string mass factor $t^{-p/2}\,\mathrm{e}^{-r^2t/2\pi\alpha'}$ to the Teichmüller integration measure, reflecting the Dirichlet nature of the $9-p$ transverse directions, where r is the separation between parallel branes. If the background field is constant, then its components which do not lie along the Dp-brane can be gauged away. The open string annulus amplitude then becomes the one-loop effective potential between two Dp-branes with generic background fields on each brane. Such a configuration describes a boundary condensate of the stretched open strings between the branes in the electromagnetic field. By keeping the transverse electric field component non-vanishing, we may also describe the interaction potential between moving branes. Much of the analysis we have made thus far for the problem of open strings in electromagnetic fields has dual analogs for D-brane dynamics. However, in the D-brane picture many of the stringy effects that we have unveiled for the external field problem have very natural dynamical explanations. In this section we will use the external field problem to describe the dynamics of D-branes. We will restrict our attention to D0-branes for simplicity.

Under T-duality, electric fields map onto the trajectories of D-branes as follows. The coupling $\exp ie \oint \vec{A}(x^0) \cdot \partial_{\parallel} \vec{x}$ of a time-varying, spatially constant electric field $\vec{E} = \partial_0 \vec{A}$ to a boundary that carries charge e is replaced by the vertex operator $\exp \frac{1}{2\pi\alpha'} \oint \vec{y}(x^0) \cdot \partial_{\perp} \vec{x}$ for a moving D0-brane [21, 38] travelling with velocity $\vec{v} = \partial_0 \vec{y} = 2\pi\alpha' e \vec{E}$. The β-function equations for this coupling can be interpreted as the classical equations of motion for the 0-brane. Constant electric field thereby corresponds to uniform motion of the branes while a neutral string would represent a pair of branes moving with zero relative velocity. In string perturbation theory the electric field and moving D-brane problems are identical because of the perturbative duality between Neumann and Dirichlet boundary conditions on the string fields [38]. In the former case the effective dynamics for a slowly varying electric field is governed by the Born-Infeld action. Under T-duality this action simply maps onto the usual action for a relativistic point particle,

$$S_{\rm D0} = \int d\tau \ T_0 \sqrt{1 - \vec{v}(\tau)^2} \,, \tag{4.1}$$

where $T_0 = 1/g_s\sqrt{\alpha'}$ is the 0-brane tension, i.e. the BPS mass of the D-particles. The Born-Infeld action is the non-trivial result of a resummation of all stringy order α' corrections, and among other things it leads to a limiting value $E_c = (2\pi\alpha' e)^{-1}$ of the external electric field, above which the system becomes unstable. In the dual picture, this is simply a consequence of the laws of relativistic particle mechanics for the 0-brane, with the critical velocity corresponding to the speed of light. At this velocity, we can make a large boost to bring the brane to rest, so that in the T-dual picture of the original string theory this amounts to boosting to large momentum. Thus the string theory with the electric background near the critical limit is equivalent to the string theory in the infinite momentum frame, or equivalently in the formalism of discrete light-cone quantization.

A planar static D-brane is a BPS defect that preserves half of the original spacetime supersymmetries. D-branes in supersymmetric configurations exert no static force on each other because supersymmetry ensures that the Casimir energy of the stretched open strings vanishes [47]. Uniform motion of a single D-brane cannot have any non-trivial consequences, because it depends on the choice of an inertial frame. However, setting a pair of branes in relative motion breaks the supersymmetry of the system and a velocity dependent potential appears between them.

Another non-BPS configuration of D-branes is that which lives in a thermal state of the theory. In the supergravity picture, non-extremal branes have a natural temperature and entropy. The Hawking radiation of a certain class of near extremal black holes with Ramond-Ramond charge may be interpreted in terms of the emission of closed string modes by a thermal state of D-branes [54, 15, 42]. It has been suggested [9] that the gravitational Hawking temperature and the temperature of a Boltzmann gas of D0-branes should be identified. This is based on the conjecture [41] that the black hole free energy resulting from classical supergravity is described accurately by the large 't Hooft coupling limit of the supersymmetric matrix quantum mechanics describing the dynamics of N D-particles [10]. In this model, which comes from the leading Yang-Mills reduction of the (non-abelian) Born-Infeld action, the brane coordinate fields are $N \times N$ hermitean matrices whose eigenvalues represent the collective coordinates of the D-particles, while the off-diagonal fluctuations represent the Higgs fields corresponding to short open string excitations between the parallel branes [58]. The breaking of supersymmetry is then explicit in the fact that the thermal partition function is computed with periodic temporal boundary conditions for the boson fields and anti-periodic ones for the fermion fields. The model accurately describes the leading velocity corrections to the tree-level action (4.1) [22], so it is natural to use it to describe the thermodynamics of moving D-branes. In this section we will describe some new calculations which compute these corrections to the static D-brane amplitudes.

4.1 Velocity dependent forces

In this subsection we will present a novel derivation, using the Polyakov path integral, of the known formula [7, 12] for the one-loop vacuum energy of D-branes moving with uniform velocity. Path integral treatments of D-branes can also be found in [32]. Consider a D-string in the presence of a boundary condensate of constant electric field E in type-IIB superstring theory. The one-loop correction to the effective Born-Infeld action at tree-level comes from the annulus string diagram which describes two D1-branes with an open string stretching between them. Neumann boundary conditions are taken along the axes 0,1, and Dirichlet ones along the transverse axes $2,\dots,d-1$. These latter directions will be labelled collectively in what follows by the superscript $\perp$. The open string parametrization is $0 \le \sigma \le 1$, $0 \le \tau \le t$, where $\ln t$ is the Teichmüller parameter of the annulus. The string carries charges e_σ at the worldsheet boundaries $\sigma = 0,1$. The euclidean action is analogous to (2.23),

$$S_{\rm D1} = S - ie_0 E \int_0^t d\tau \; x^1 \, \partial_\tau x^0 \Big|^{\sigma=0} + ie_1 E \int_0^t d\tau \; x^1 \, \partial_\tau x^0 \Big|^{\sigma=1} \,, \tag{4.2}$$

where the bulk action S is given by

$$S = \frac{1}{4\pi\alpha'} \int_0^t d\tau \int_0^1 d\sigma \; \left(\partial_\tau x^\mu \, \partial_\tau x_\mu + \partial_\sigma x^\mu \, \partial_\sigma x_\mu\right). \tag{4.3}$$

Varying the action (4.2) leads to the boundary conditions

$$\begin{aligned}
&\left(\partial_\sigma x^0 - 2\pi i\alpha' e_0 E \, \partial_\tau x^1\right)_{\sigma=0} = 0\,, \\
&\left(\partial_\sigma x^0 - 2\pi i\alpha' e_1 E \, \partial_\tau x^1\right)_{\sigma=1} = 0\,, \\
&\left(\partial_\sigma x^1 - 2\pi i\alpha' e_0 E \, \partial_\tau x^0\right)_{\sigma=0} = 0\,, \\
&\left(\partial_\sigma x^1 - 2\pi i\alpha' e_1 E \, \partial_\tau x^0\right)_{\sigma=1} = 0\,, \\
&\partial_\tau x^\perp\Big|_{\sigma=0} = 0 \;\text{ or }\; x^\perp\Big|_{\sigma=0} = y_0^\perp\,, \\
&\partial_\tau x^\perp\Big|_{\sigma=1} = 0 \;\text{ or }\; x^\perp\Big|_{\sigma=1} = y_1^\perp\,.
\end{aligned} \tag{4.4}$$

4.1.1 Bosonic case

If the coordinate $x^1 \equiv x^1_{\rm N}$ is compactified on a circle of circumference $L_{\rm N}$, then we can make a T-duality transformation along the 1-axis which interchanges the Neumann and Dirichlet boundary conditions for the open string. The new coordinate $x^1_{\rm D}$ takes values on a dual circle of circumference $L_{\rm D} = 4\pi^2\alpha'/L_{\rm N}$ with $\partial_\tau x^1_{\rm N} = \partial_\sigma x^1_{\rm D}$ and $\partial_\sigma x^1_{\rm N} = \partial_\tau x^1_{\rm D}$. The boundary conditions (4.4) can be rewritten as

$$\begin{aligned}
&\partial_\sigma \left(x^0 - u_0 x^1\right)\Big|_{\sigma=0} = 0\,,\\
&\partial_\sigma \left(x^0 - u_1 x^1\right)\Big|_{\sigma=1} = 0\,,\\
&\partial_\tau \left(x^1 + u_0 x^0\right)\Big|_{\sigma=0} = 0\,,\\
&\partial_\tau \left(x^1 + u_1 x^0\right)\Big|_{\sigma=1} = 0\,,\\
&\partial_\tau x^\perp\Big|_{\sigma=0} = 0 \quad \text{or} \quad x^\perp\Big|_{\sigma=0} = y_0^\perp\,,\\
&\partial_\tau x^\perp\Big|_{\sigma=1} = 0 \quad \text{or} \quad x^\perp\Big|_{\sigma=1} = y_1^\perp\,,
\end{aligned} \tag{4.5}$$

where we have simply denoted $x^1 \equiv x^1_{\rm D}$, and

$$u_\sigma = iv_\sigma = 2\pi i\alpha' e_\sigma E\,, \qquad \sigma = 0,1 \tag{4.6}$$

are the analytic continuations of the velocities of the two string endpoints from Minkowski to euclidean space arising from Wick rotation of both x^0 and τ.

In the case of static D0-branes ($v_\sigma = 0$), the mode expansions of the string fields which diagonalize the action and solve the boundary conditions are well-known to be given by [43]

$$\begin{aligned}
x^0_{(0)}(\tau,\sigma) &= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} a_{mn}\,{\rm e}^{\,2\pi i m\tau/t}\,\cos\pi n\sigma\,,\\
x^1_{(0)}(\tau,\sigma) &= y_0^1 + \left(y_1^1 - y_0^1\right)\sigma \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} b_{mn}\,{\rm e}^{\,2\pi i m\tau/t}\,\sin\pi n\sigma\,,\\
x^\perp_{(0)}(\tau,\sigma) &= y_0^\perp + \left(y_1^\perp - y_0^\perp\right)\sigma + \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} x^\perp_{mn}\,{\rm e}^{\,2\pi i m\tau/t}\,\sin\pi n\sigma\,,
\end{aligned} \tag{4.7}$$

where $a_{-mn} = a^*_{mn}$, $b_{-mn} = b^*_{mn}$, and $x^\perp_{-mn} = (x^\perp_{mn})^*$. The mode expansions which solve the boundary conditions (4.5) may then be obtained by rotating the fields (4.7) through angle $\pi\alpha_0 + \pi\alpha\sigma$ in the 0–1 plane to get

$$\begin{aligned}
x^0(\tau,\sigma) &= \cos\left(\pi\alpha_0 + \pi\alpha\sigma\right) x^0_{(0)}(\tau,\sigma) + \sin\left(\pi\alpha_0 + \pi\alpha\sigma\right) x^1_{(0)}(\tau,\sigma)\,,\\
x^1(\tau,\sigma) &= -\sin\left(\pi\alpha_0 + \pi\alpha\sigma\right) x^0_{(0)}(\tau,\sigma) + \cos\left(\pi\alpha_0 + \pi\alpha\sigma\right) x^1_{(0)}(\tau,\sigma)\,,\\
x^\perp(\tau,\sigma) &= x^\perp_{(0)}(\tau,\sigma)\,.
\end{aligned} \tag{4.8}$$

The fields (4.8) obey the boundary conditions (4.5) with the identifications of the rotation angles as the rapidities of the euclidean boost,

$$\pi\alpha_\sigma = \arctan u_\sigma\,, \qquad \sigma = 0,1\,, \qquad \alpha = \alpha_1 - \alpha_0\,. \tag{4.9}$$

The mode expansion (4.8) diagonalizes the action (4.3) which enters the euclidean path integral

$$\mathcal{F} = 2\,\frac{(2\alpha')^{-d/2}}{\pi}\int_0^\infty \frac{dt}{t}\int Dx^\mu\ D({\rm ghosts})\,{\rm e}^{\,-S-S_{\rm ghosts}} \tag{4.10}$$

with the boundary conditions (4.5). Here an extra factor of 2 has been inserted to account for the symmetry under interchange of the two endpoints of an oriented string [47].

Let us begin by evaluating the contributions from the non-zero modes of the fields x^0 and x^1 to the path integral (4.10). The modes with indices (m,n) and (m',n') are orthogonal for $m' \neq -m$, and so their contribution to the action (4.3) is

$$S_{\rm q} = \frac{\pi t}{8\alpha'} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \Bigg[\left(\alpha^2 + n^2 + \frac{4m^2}{t^2} \right) \Big(a_{mn} a_{-mn} + b_{mn} b_{-mn} \Big) - \\ - 2\alpha n \Big(a_{mn} b_{-mn} + a_{-mn} b_{mn} \Big) \Bigg] , \qquad (4.11)$$

where we have used the fact that the modes with either sine or cosine functions are orthogonal over the semi-period of σ, while the cross terms which mix sine and cosine functions do not occur. Evaluating the resulting gaussian functional integral in (4.10) over these non-zero modes produces the determinant

$$\prod_{m,n} \begin{vmatrix} (\alpha^2+n^2)t + \frac{4m^2}{t} & -2\alpha n t \\ -2\alpha n t & (\alpha^2+n^2)t + \frac{4m^2}{t} \end{vmatrix}^{-1/2} = \prod_{m=-\infty}^{\infty} \prod_{n=1}^{\infty} \Big((n+\alpha)\sqrt{t} + 2im/\sqrt{t} \Big)^{-1} \times \\ \times \Big((n-\alpha)\sqrt{t} + 2im/\sqrt{t} \Big)^{-1} \\ = \prod_{n=1}^{\infty} \frac{1}{4 \sinh \frac{\pi(n+\alpha)t}{2} \, \sinh \frac{\pi(n-\alpha)t}{2}} , \qquad (4.12)$$

where we have used the formula

$$\prod_{m=-\infty}^{\infty} \Big(my + x \Big) = \sin \frac{\pi x}{y} . \qquad (4.13)$$

In arriving at (4.12) we have ignored an overall constant factor which may be set equal to unity by using zeta-function regularization (2.12) of the infinite product.

Analogously, the contribution to (4.3) from the zero modes of the fields x^0 and x^1 is

$$S_0 = \frac{\pi t}{4\alpha'} \sum_{m=-\infty}^{\infty} \left(\alpha^2 + \frac{4m^2}{t^2} \right) a_{m0}\, a_{-m0} - \frac{\alpha t}{2\alpha'} a_{00} \left(y_1^1 - y_0^1 \right) + \frac{t}{4\pi\alpha'} \left(y_1^1 - y_0^1 \right)^2 . \qquad (4.14)$$

The corresponding gaussian functional integral gives

$$\prod_{m=-\infty}^{\infty} \left(\alpha^2 t + \frac{4m^2}{t} \right)^{-1/2} = \frac{1}{2 \sinh \frac{\pi \alpha t}{2}} . \qquad (4.15)$$

For any non-vanishing α the distance $y_1^1 - y_0^1$ between the ends of the string along the direction of motion can be absorbed into the quantity a_{00} and disappears from the final result, as it should. If $\alpha = 0$ the integral over a_{00} produces the volume along the 0-axis and the last term on the right-hand side of eq. (4.14) remains just as for the transverse directions which contribute the usual quantity

$$\prod_{n=1}^{\infty} \frac{1}{\left(2 \sinh \frac{\pi n t}{2} \right)^{d-2}} \mathrm{e}^{-\frac{t}{4\pi\alpha'} (y_1^\perp - y_0^\perp)^2} = \frac{1}{\eta \left(\frac{it}{2} \right)^{d-2}} \mathrm{e}^{-\frac{t}{4\pi\alpha'} (y_1^\perp - y_0^\perp)^2} . \qquad (4.16)$$

We now multiply the three quantities (4.12), (4.15) and (4.16) together, take into account the contributions from the conformal ghost fields, and use the identity

$$\frac{1}{2\sinh\frac{\pi\alpha t}{2}}\prod_{n=1}^{\infty}\frac{\sinh^2\frac{\pi n t}{2}}{\sinh\frac{\pi(n+\alpha)t}{2}\,\sinh\frac{\pi(n-\alpha)t}{2}}=\frac{1}{2\pi}\,\frac{\Theta_1'\left(0\,\middle|\,\frac{it}{2}\right)}{\Theta_1\left(\frac{i\alpha t}{2}\,\middle|\,\frac{it}{2}\right)}\,. \tag{4.17}$$

In this way we find that the vacuum energy functional (4.10) is given by

$$\mathcal{F}=\frac{1}{2\pi}\int_0^\infty\frac{dt}{t}\;\mathrm{e}^{-\frac{t}{4\pi\alpha'}(y_1^\perp-y_0^\perp)^2}\,\frac{\Theta_1'\left(0\,\middle|\,\frac{it}{2}\right)}{\Theta_1\left(\frac{i\alpha t}{2}\,\middle|\,\frac{it}{2}\right)}\,\frac{1}{\eta\left(\frac{it}{2}\right)^{24}} \tag{4.18}$$

in the critical dimension $d=26$. This coincides with the result of refs. [7, 12] for the bosonic string which is expressed in terms of the Minkowski space rapidity $\epsilon=i\alpha$. Note that, in the present derivation, we have simply used the action (4.3) without explicitly adding boundary terms to correctly reproduce the bosonic amplitude for moving D0-branes.

4.1.2 RNS formulation

We will now turn to the superstring vacuum amplitude. The open string is again parametrized by the worldsheet coordinates (τ,σ). On the annulus there are four standard spin structures, $(+,-)$, $(+,+)$ in the R-sector (associated with spacetime fermions) and $(-,-)$, $(-,+)$ in the NS-sector (associated with spacetime bosons), which represent the periodicities of the worldsheet fermion fields with respect to (τ,σ). The GSO projection dictates that the physical amplitude is obtained by summing over the contributions from the four spin structures. The fermionic part of the superstring action is given by

$$S_{\rm F}=\frac{i}{2}\int_0^t d\tau\int_0^1 d\sigma\,\Big(\psi^\mu(\partial_\tau+i\partial_\sigma)\psi^\mu\tilde\psi^\mu(\partial_\tau-i\partial_\sigma)\tilde\psi^\mu\Big) \tag{4.19}$$

where ψ^μ and $\tilde\psi^\mu$ are Grassmann-valued fields which transform as SO(8) vectors.

In the absence of an external field, or for static D-branes, the fermion fields obey the standard superstring boundary conditions

$$\begin{aligned}\psi^\mu_{(0)}(\tau,0)&=\tilde\psi^\mu_{(0)}(\tau,0)\,,\\ \psi^\mu_{(0)}(\tau,1)&=(-1)^a\,\tilde\psi^\mu_{(0)}(\tau,1)\,,\end{aligned} \tag{4.20}$$

where $a=0$ for the R-sector and $a=1$ for the NS-sector. The mode expansions in the $(\pm,-)$ sectors are given by

$$\begin{aligned}\psi^\mu_{(0)}(\tau,\sigma)&=\sum_{m,n=-\infty}^{\infty}\psi^\mu_{mn}\;\mathrm{e}^{\pi i(2m+1)\tau/t}\;\mathrm{e}^{\pi i(n+\frac{a}{2})\sigma}\,,\\ \tilde\psi^\mu_{(0)}(\tau,\sigma)&=\sum_{m,n=-\infty}^{\infty}\psi^\mu_{mn}\;\mathrm{e}^{\pi i(2m+1)\tau/t}\;\mathrm{e}^{-\pi i(n+\frac{a}{2})\sigma}\,,\end{aligned} \tag{4.21}$$

while in the $(\pm,+)$ sectors they are

$$\begin{aligned}\psi^\mu_{(0)}(\tau,\sigma)&=\sum_{m,n=-\infty}^{\infty}\psi^\mu_{mn}\;\mathrm{e}^{2\pi i m\tau/t}\;\mathrm{e}^{\pi i(n+\frac{a}{2})\sigma}\,,\\ \tilde\psi^\mu_{(0)}(\tau,\sigma)&=\sum_{m,n=-\infty}^{\infty}\psi^\mu_{mn}\;\mathrm{e}^{2\pi i m\tau/t}\;\mathrm{e}^{-\pi i(n+\frac{a}{2})\sigma}\,.\end{aligned} \tag{4.22}$$

For moving branes we rotate the fields $\psi^\mu_{(0)}$ with respect to $\tilde\psi^\mu_{(0)}$ by the orthogonal matrix [29]

$$M_{\mu\nu}(\sigma) = \mathrm{e}^{\,(\pi\alpha_0+\pi\alpha\sigma)\Sigma^{01}_{\mu\nu}} \tag{4.23}$$

where $\Sigma^{\rho\lambda}_{\mu\nu} = \delta^\rho_\mu\,\delta^\lambda_\nu - \delta^\lambda_\mu\,\delta^\rho_\nu$ are the generators of SO(8) rotations in the vector representation and the rapidities α are given by eq. (4.9). Since

$$\begin{aligned} M_{\mu\nu}(0) &= \mathrm{e}^{\,\pi\alpha_0\Sigma^{01}_{\mu\nu}} = \left(\frac{\mathbb{1}+u_0\Sigma^{01}}{\mathbb{1}-u_0\Sigma^{01}}\right)_{\mu\nu}\,, \\ M_{\mu\nu}(1) &= \mathrm{e}^{\,\pi\alpha_1\Sigma^{01}_{\mu\nu}} = \left(\frac{\mathbb{1}+u_1\Sigma^{01}}{\mathbb{1}-u_1\Sigma^{01}}\right)_{\mu\nu}\,, \end{aligned} \tag{4.24}$$

the rotation of the fermion fields by (4.23) solves the boundary conditions

$$\begin{aligned} \psi_\mu(\tau,0) - \tilde\psi_\mu(\tau,0) &= u_0\Sigma^{01}_{\mu\nu}\left(\psi^\nu(\tau,0)+\tilde\psi^\nu(\tau,0)\right), \\ \psi_\mu(\tau,1) - (-1)^a\,\tilde\psi_\mu(\tau,1) &= u_1\Sigma^{01}_{\mu\nu}\left(\psi^\nu(\tau,1)+(-1)^a\,\tilde\psi^\nu(\tau,1)\right), \end{aligned} \tag{4.25}$$

which extend (4.20) to finite velocity provided that (4.9) is satisfied. The mode expansions are thereby given from (4.21), (4.22) as

$$\begin{aligned} \psi_\mu(\tau,\sigma) &= M_{\mu\nu}(\sigma)^{1/2}\,\psi^\nu_{(0)}(\tau,\sigma)\,, \\ \tilde\psi_\mu(\tau,\sigma) &= M_{\mu\nu}(\sigma)^{-1/2}\,\tilde\psi^\nu_{(0)}(\tau,\sigma)\,, \end{aligned} \tag{4.26}$$

where here only the 0 and 1 components of the fields are rotated since the boost is in the 0–1 plane. The mode expansions (4.26) diagonalize the action (4.19) in each of the four worldsheet sectors. We shall analyse the contributions first from each sector separately.

$(+,-)$ **sector:** using (4.21) with $a=0$ and substituting (4.26) into (4.19) leads to the action

$$S_{\rm F} = -\pi\sum_{m=-\infty}^\infty\,\sum_{n=-\infty}^\infty \psi^\mu_{-m-n}\left(2m+1+t(in+\alpha\Sigma^{01})\right)_{\mu\nu}\psi^\nu_{mn}\,. \tag{4.27}$$

The functional gaussian integral over the Grassmann variables ψ^μ_{mn} thereby produces the determinant

$$\begin{aligned} &\prod_{m,n=-\infty}^\infty \det\begin{pmatrix} 2m+1+int & \alpha t \\ -\alpha t & 2m+1+int\end{pmatrix}^{1/2}\left(2m+1+int\right)^4 = \\ &\quad= \prod_{m=-\infty}^\infty\,\prod_{n=-\infty}^\infty\left(2m+1+it(n+\alpha)\right)^{1/2}\times \\ &\qquad\times\left(2m+1+it(n-\alpha)\right)^{1/2}\left(2m+1+int\right)^4 \\ &\quad= 2^5\cosh\frac{\pi t\alpha}{2}\prod_{n=1}^\infty 4\cosh\frac{\pi t(n+\alpha)}{2}\cosh\frac{\pi t(n-\alpha)}{2}\left(2\cosh\frac{\pi tn}{2}\right)^8 \\ &\quad= \frac{\Theta_2\left(\frac{i\alpha t}{2}\,\middle|\,\frac{it}{2}\right)}{\eta\left(\frac{it}{2}\right)}\left(\frac{\Theta_2\left(0\,\middle|\,\frac{it}{2}\right)}{\eta\left(\frac{it}{2}\right)}\right)^4, \end{aligned} \tag{4.28}$$

where we have used the formula

$$\cos \pi x = \prod_{k=0}^{\infty} \left[1 - \frac{4x^2}{(2k+1)^2} \right] . \tag{4.29}$$

$(+,+)$ sector: for $a = 0$ the fermion fields (4.22) have a zero mode, and so the contribution from this sector vanishes as usual due to the integration over the fermionic zero mode.

$(-,-)$ sector: setting $a = 1$ in (4.21) and substituting (4.26) into (4.19) leads to the action

$$S_{\mathrm{F}} = -\pi \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \psi^{\mu}_{-m-n} \left[2m + 1 + t \left(i \left(n + \frac{1}{2} \right) + \alpha \Sigma^{01} \right) \right]_{\mu\nu} \psi^{\nu}_{mn} . \tag{4.30}$$

The functional gaussian integral over the Grassmann variables ψ^{μ}_{mn} gives

$$\begin{aligned} \prod_{m=-\infty}^{\infty} \prod_{n=-\infty}^{\infty} &\Big(2m + 1 + it\Big(n + \frac{1}{2} + \alpha\Big)\Big)^{1/2} \Big(2m + 1 + it\Big(n + \frac{1}{2} - \alpha\Big)\Big)^{1/2} \times \\ &\qquad \times \Big(2m + 1 + it\Big(n + \tfrac{1}{2}\Big)\Big)^{4} = \\ &= \prod_{n=1}^{\infty} 4 \cosh \frac{\pi t (n - \frac{1}{2} + \alpha)}{2} \cosh \frac{\pi t (n - \frac{1}{2} - \alpha)}{2} \left(2 \cosh \frac{\pi t (n - \frac{1}{2})}{2} \right)^{8} \\ &= \frac{\Theta_3 \left(\frac{i\alpha t}{2} \,\middle|\, \frac{it}{2} \right)}{\eta \left(\frac{it}{2} \right)} \left(\frac{\Theta_3 \left(0 \,\middle|\, \frac{it}{2} \right)}{\eta \left(\frac{it}{2} \right)} \right)^{4} . \end{aligned} \tag{4.31}$$

$(-,+)$ sector: finally, by setting $a = 1$ in (4.22) the action reads

$$S_{\mathrm{F}} = -\pi \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \psi^{\mu}_{-m-n} \left[2m + t \left(i \left(n + \frac{1}{2} \right) + \alpha \Sigma^{01} \right) \right]_{\mu\nu} \psi^{\nu}_{mn} . \tag{4.32}$$

The functional gaussian integral over the Grassmann variables ψ^{μ}_{mn} yields

$$\begin{aligned} \prod_{m=-\infty}^{\infty} \prod_{n=-\infty}^{\infty} &\Big(2m + it\Big(n + \frac{1}{2} + \alpha\Big)\Big)^{1/2} \Big(2m + it\Big(n + \frac{1}{2} - \alpha\Big)\Big)^{1/2} \Big(2m + it\Big(n + \frac{1}{2}\Big)\Big)^{4} = \\ &= \prod_{n=1}^{\infty} 4 \sinh \frac{\pi t (n - \frac{1}{2} + \alpha)}{2} \sinh \frac{\pi t (n - \frac{1}{2} - \alpha)}{2} \left(2 \sinh \frac{\pi t (n - \frac{1}{2})}{2} \right)^{8} \\ &= \frac{\Theta_4 \left(\frac{i\alpha t}{2} \,\middle|\, \frac{it}{2} \right)}{\eta \left(\frac{it}{2} \right)} \left(\frac{\Theta_4 \left(0 \,\middle|\, \frac{it}{2} \right)}{\eta \left(\frac{it}{2} \right)} \right)^{4} . \end{aligned} \tag{4.33}$$

We now take into account the contributions from the conformal anti-ghost fields in each of the three non-vanishing sectors above, sum over the spin structures with weight $1/2$, and multiply by the bosonic amplitude in the superstring critical dimension $d = 10$. In this way we arrive at the known formula [7] for the superstring vacuum energy functional,

$$
\begin{aligned}
\mathcal{F} = \frac{1}{2\pi} \int_0^\infty \frac{dt}{t}\, \mathrm{e}^{-\frac{t}{4\pi\alpha'}(y_1^\perp - y_0^\perp)^2}\, \frac{\Theta_1'\left(0\,\big|\,\frac{it}{2}\right)}{\Theta_1\left(\frac{i\alpha t}{2}\,\big|\,\frac{it}{2}\right)}\, \frac{1}{\eta\left(\frac{it}{2}\right)^{12}} \\
\times \frac{1}{2}\Bigg[\Theta_3\left(\frac{i\alpha t}{2}\bigg|\frac{it}{2}\right)\Theta_3\left(0\bigg|\frac{it}{2}\right)^3 - \Theta_4\left(\frac{i\alpha t}{2}\bigg|\frac{it}{2}\right)\Theta_4\left(0\bigg|\frac{it}{2}\right)^3 - \\
- \Theta_2\left(\frac{i\alpha t}{2}\bigg|\frac{it}{2}\right)\Theta_2\left(0\bigg|\frac{it}{2}\right)^3\Bigg]. \qquad (4.34)
\end{aligned}
$$

By using the formula

$$
\frac{1}{2}\left[\Theta_3(\nu|it)\,\Theta_3(0|it)^3 - \Theta_4(\nu|it)\,\Theta_4(0|it)^3 - \Theta_2(\nu|it)\,\Theta_2(0|it)^3\right] = \Theta_1\left(\frac{\nu}{2}\bigg|it\right)^4 \qquad (4.35)
$$

which is a consequence of the Riemann identity for Jacobi theta-functions, we arrive at the result of refs. [29, 12] after a modular transformation $t \mapsto 1/t$ of the annular Teichmüller parameter.

4.2 D-brane scattering

After analytical continuation to Minkowski space, the quantity (4.34) can be interpreted as the forward scattering amplitude for two D-particles moving with relative velocity $v = \tanh \pi\epsilon$ and impact parameter $b = |y_1^\perp - y_0^\perp|$. It is a semi-classical result, in that the D-branes are treated as classical sources and higher worldsheet topologies are neglected, i.e. both the Compton wavelength and the Schwarzschild radius of the D-branes are taken to vanish. The branes interact via virtual pairs of open oriented strings which are stretched by the relative motion. The integrand of (4.34) has an infinite number of poles along the real t-axis at $t = 2\pi(2n+1)/\epsilon$, where n is an integer. These poles arise from the zeroes of the trigonometric sine function in the product representation of the theta-function Θ_1. As a consequence, the vacuum energy acquires an imaginary part which is given by the sum over the residues of the poles and which gives the probability that the virtual strings materialize. This phenomenon is simply the dual counterpart of the open string pair production in a uniform background electric field that we described in section 2.2. In the present case the result has a much simpler interpretation. As the two D-particles move away from each other, they continuously transfer their energy to any open strings that stretch between them. A virtual pair of open strings can therefore nucleate out of the vacuum and slow down or even completely stop the relative motion.

The real part of the scattering amplitude also reveals a striking feature of the low velocity dynamics of D-particles. The theta-functions $\Theta_a(\nu|\tau)$ are even functions of ν, and in the low velocity limit $\Theta_1(\frac{\epsilon t}{2}|\frac{it}{2})^4 \sim \mathcal{O}(v^4)$. The absence of a constant term in the velocity expansion of (4.34) is due to the cancellation of the gravitational attraction and the Ramond-Ramond repulsion for static D-branes [47]. However, not only the static, but also the order v^2 force between two D-particles vanishes, i.e. identical type-II D-branes do not scatter at non-relativistic velocities. Generally, the order v^2 scattering of heavy solitons can be described by geodesic motion in the moduli space of zero modes. Therefore, the moduli space of a pair of D-particles, which at tree level is the flat quotient space $\mathbb{R}^9 \times \mathbb{R}^9/\mathbb{Z}_2$, remains completely flat to all orders in the α' expansion. This is the dual statement of the fact that for maximally supersymmetric gauge theories the Maxwell F^2 term in the effective action is not renormalized. The next contribution comes at order v^4 in the velocity expansion. The expansion of the effective action for two D-particles in their velocities divided by powers of their separation r is thereby given as

$$S_{\mathrm{D0}} = \int d\tau \left[\frac{T_0}{2} \left((\dot{y}_0^\perp)^2 + (\dot{y}_1^\perp)^2 \right) \frac{15}{16} \, (\alpha')^3 \, \frac{v^4}{r^7} + \cdots \right] . \tag{4.36}$$

The v^4 potential in (4.36) is the standard interaction term for D0-branes in ten dimensional supergravity [22].

The vacuum energy functional (4.34) could actually have been determined using standard formulas for the partition functions of free massless fields with twisted boundary conditions. The spectrum of an open string stretched between two moving D-branes can be determined from the operatorial mode expansions of the light-cone fields $x^\pm = \frac{1}{\sqrt{2}}(x^0 \mp x^1)$ which are given by

$$x^\pm(\tau,\sigma) = -i \sqrt{\frac{\alpha'}{2}} \sqrt{\frac{1 \pm v_0}{1 \mp v_0}} \sum_{n=-\infty}^{\infty} \left(\frac{a_n^\pm}{n \pm i\epsilon} \mathrm{e}^{\,-\pi i (n \pm i\epsilon)(\tau+\sigma)} + \frac{a_n^\mp}{n \mp i\epsilon} \mathrm{e}^{\,-\pi i (n \mp i\epsilon)(\tau-\sigma)} \right) . \tag{4.37}$$

It is easy to verify that x^0 and x^1 then obey the boundary conditions (4.5). In particular, $x^\pm(\tau,1) = \mathrm{e}^{\,\pm\pi\epsilon}\, x^\pm(\tau,0)$, so that the two string endpoints have relative velocity $v = \tanh \pi\epsilon$. Reality requires $(a_n^\pm)^\dagger = a_{-n}^\pm$, while canonical quantization implies the commutation relations $[a_n^+, a_m^-] = (n + i\epsilon)\, \delta_{n+m,0}$. The D-brane motion modifies the vacuum energy as can be read off from the light-like component of the total worldsheet hamiltonian

$$L_0^{||} = \frac{b^2}{4\pi^2 \alpha'} + \sum_{n=1}^{\infty} (a_n^+)^\dagger a_n^- + \sum_{n=0}^{\infty} (a_n^-)^\dagger a_n^+ + \frac{i\epsilon(1 - i\epsilon)}{2} \, , \tag{4.38}$$

where we have included the dependence on the impact parameter. Analogous mode expansions arise for the worldsheet fermion fields. The relevant effect of the brane motion on the stretched strings is to shift their oscillation frequencies by $\pm i\epsilon$ in the boost plane and their energy by an overall velocity dependent term. Similar expansions arise in the twisted sectors of orbifold conformal field theories, with $i\epsilon$ identified as a real-valued rotation angle. Therefore, in the operator formalism, the problem of moving D-branes is formally identical to that of the stretched strings between the branes belonging to a twisted sector of an orbifold string theory with imaginary twist angle corresponding to the rapidity of the boost.

All of this is again completely analogous to the spectrum of free open strings in a uniform electric field background, except for some important changes. The expression (4.9) for the twist parameter $\epsilon = i\alpha$ has no obvious interpretation in the electric field case, while here it is recognized as the relativistic sum of the two brane velocities. Consistent with Lorentz invariance, the spectrum only depends on the velocity v of one brane in the rest frame of the other. Furthermore, zero modes are omitted in the light-cone mode expansions (4.37) to account for the fact that D-branes interact locally in transverse space and in time. Keeping this and the orbifold analogy in mind, it is straightforward to arrive at the annulus amplitude (4.34). The orbifold interpretation further enables a very simple calculation of the velocity dependent potential in (4.36). In the quasi-static approximation, which is valid to leading order in the inverse separation, the potential is given simply as the sum of the ground state energies of the corresponding harmonic oscillators as in (4.38). There are ten complex bosonic oscillators of frequencies $\omega_{\rm b}^{\perp} = \sqrt{r^2}$ with multiplicity eight and $\omega_{\rm b}^{\pm} = \sqrt{r^2 \pm 2iv}$ with multiplicity one each, where $r = \sqrt{b^2 + v^2\tau^2}$ is the distance between the branes at time τ. There are also two ghost oscillators each of frequency $\omega_{\rm ghosts} = \sqrt{r^2}$, and 16 fermionic oscillators of frequencies $\omega_{\rm f}^{\pm} = \sqrt{r^2 \pm iv}$ with multiplicity eight each. The velocity dependent potential is then given by

$$V(r) = 8\,\omega_{\rm b}^{\perp} + \omega_{\rm b}^{+} + \omega_{\rm b}^{-} - 2\,\omega_{\rm ghosts} - 4\,\omega_{\rm f}^{+} - 4\,\omega_{\rm f}^{-}\ . \tag{4.39}$$

For $v = 0$ the frequencies cancel and the static potential vanishes. For $v \neq 0$ we can expand each frequency as a power series in r^{-1}. At the first three orders in v/r^2 the potential vanishes, while at fourth order the energy between the 0-branes gives the expected leading result $V(r) = -15v^4/16r^7 + \cdots$.

4.3 Thermodynamics

Just as we did with the thermal configuration of free open strings in background electric fields, it is possible to demonstrate that there are no excited states of a pair of moving D-branes with $v \neq 0$ at finite temperature [3]. Again the partition function picks up only the temperature independent piece, and we may conclude that D-brane dynamics forbid uniform velocity motion at finite temperature. The triviality comes from the same zero mode operators, associated with the presence of a Wu-Yang term, as in the electric field problem. Using T-duality, we may therefore attribute this property of D-brane dynamics to the Debye screening of electric fields that we discussed in section 3.2.2. Just as Debye screening forbids constant electric fields in open superstring theory, it also forbids the uniform motion of D-branes. This implies that there is a damping of their motion analogous to Debye screening. Recall that, in the dual picture, this is not the case for constant *magnetic* fields. At one-loop order this corresponds to a relative disalignment between a pair of branes, which is an allowed configuration at finite temperature.

To investigate further the properties of D-brane dynamics at finite temperature, one must consider appropriate non-uniform motion. This is a difficult problem to treat using the usual, direct methods of string perturbation theory, as is the dual problem for time-dependent background fields. However, one can compute the thermodynamic free energy for moving D0-branes by using the low-energy effective Yang-Mills theory description of the D-brane dynamics [58, 10]. Then, a perturbative calculation will be valid in the domain where $g_s^{1/3}\sqrt{\alpha'} \ll r$. We can therefore effectively describe the thermodynamics in the limit of weak string coupling, or equivalently when the branes are well separated. Since the D0-branes have mass $T_0 = 1/g_s\sqrt{\alpha'}$ and are therefore very heavy in this limit, this calculation will take into account the thermal fluctuations of the stretched superstrings but not of the D-particles themselves.

The action is obtained from the dimensional reduction of ten-dimensional maximally supersymmetric Yang-Mills theory to one temporal and zero spatial dimensions,

$$S_{\rm YM}[A,\Psi]=\frac{1}{g_{\rm YM}^2}\int d\tau\ {\rm tr}\left(\frac{1}{4}\,F_{\mu\nu}^2+\frac{i}{2}\,\Psi\gamma^\mu D_\mu\Psi\right) \tag{4.40}$$

where the Yang-Mills coupling constant $g_{\rm YM}$ is related to the string coupling g_s by $g_{\rm YM}^2 = g_s/4\pi^2(\alpha')^{3/2}$. The gauge fields $A_\mu(\tau)$ and the Majorana spinor fields $\Psi(\tau)$ depend only on the time coordinate τ. The diagonal components

$$\vec{y}^{\,b}=2\pi\alpha'\vec{a}^{\,b}\equiv 2\pi\alpha'\vec{A}^{\,bb} \tag{4.41}$$

of the gauge fields are interpreted as the position of the b-th D0-brane and are treated as collective variables. The thermal partition function defines the statistical mechanics of the gas of D0-branes through the path integral

$$Z_{\rm YM}=\int D\vec{y}^{\,a}\ {\rm e}^{\,-S_{\rm eff}[\vec{y}^{\,a}]} \tag{4.42}$$

with the euclidean time coordinate τ compactified on a circle of circumference $\beta=1/k_{\rm B}T$. The effective action for the D-particle coordinates is constructed by integrating out the off-diagonal components of the gauge fields, the fermion fields, and the Faddeev-Popov ghost fields required for gauge fixing,

$$S_{\rm eff}[\vec{y}^{\,a}]=-\ln\int\prod_{a\neq b}Da_0^b\ DA_\mu^{ab}\int D\Psi D({\rm ghosts}){\rm e}^{\,-S_{\rm YM}-S_{\rm ghosts}}\,, \tag{4.43}$$

with periodic boundary conditions for the gauge and ghost fields and anti-periodic ones for the adjoint fermion fields around the compact temperature direction. We will consider again only the case of a single pair of D0-branes whose worldlines lie along the periodic temporal direction.

The integration in (4.43) can be done in a simultaneous loop expansion in the gauge theory and in a velocity expansion in the brane configurations. This will produce meaningful results in the limit where $r\equiv|\vec{y}^{\,1}-\vec{y}^{\,2}|$ is large and where $\vec{v}^{\,a}=\dot{\vec{y}}^{\,a}$ are small (Dot denotes differentiation with respect to τ). The one-loop contribution can be obtained by expanding the action (4.40) to second order in the off-diagonal components of the gauge fields, and in the ghost and fermion fields. The result of this standard gaussian functional integration produces a ratio of determinants of the form [2]

$$\begin{aligned}S_{\rm eff}\ =&\int_0^\beta d\tau\ \frac{1}{4g_{\rm YM}^2}\left((\dot{\vec{a}}^{\,1})^2+(\dot{\vec{a}}^{\,2})^2\right)+\frac{1}{2}\ {\rm tr}_B\ln\Big[\delta_{\mu\nu}\,\mathcal{D}^2+2i(f_{\mu\nu}^1-f_{\mu\nu}^2)\Big]+\\ &+\frac{1}{2}\ {\rm tr}_B\ln\Big[\delta_{\mu\nu}\,\mathcal{D}^2-2i(f_{\mu\nu}^1-f_{\mu\nu}^2)\Big]-2\,{\rm tr}_B\ln\mathcal{D}^2-8\,{\rm tr}_F\ln\mathcal{D}^2\,,\end{aligned} \tag{4.44}$$

where we have introduced the second order differential operator on the temperature circle,

$$\mathcal{D}^2=(-i\partial_\tau+a_0)^2+|\vec{a}|^2\,, \tag{4.45}$$

which arises from the gauge covariant derivatives. Here $a_\mu = a^1_\mu - a^2_\mu$, and we have included the tree-level term which gives the non-relativistic kinetic energies of the D0-branes. The subscript B (resp. F) indicates that the determinant is to be evaluated with periodic (resp. anti-periodic) boundary conditions corresponding to the contributions from the gauge and ghost (resp. adjoint fermion) fields, respectively. The abelian field strength tensors $f^b_{\mu\nu}$ have non-vanishing components $f^b_{0i} = \dot{a}^b_i$. The temporal components a^b_0 of the gauge fields may be taken to be independent of the compactified time variable via the residual abelian gauge invariance of the problem, and by periodicity to lie in the interval $(-\frac{\pi}{\beta}, \frac{\pi}{\beta}]$. The determinants in (4.44) have been evaluated for static D-brane configurations in [2]. In what follows we shall extend this computation to leading orders in the velocity expansion for moving D-branes.

By using the proper time representation (2.34) we are led to first evaluate the trace

$$\operatorname{tr}\, \mathrm{e}^{-t\mathcal{D}^2} = \int_0^\beta d\tau \lim_{\tau'\to\tau} \mathrm{e}^{-t\mathcal{D}^2}\, \delta(\tau-\tau')\,. \tag{4.46}$$

We will begin by computing the determinant in (4.44) with bosonic boundary conditions on the temporal circle. For this, we insert the periodic delta-function

$$\delta_B(\tau-\tau') = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi i n(\tau-\tau')/\beta} \tag{4.47}$$

into (4.46), which incorporates the proper Matsubara frequencies and gives

$$\operatorname{tr}_B\, \mathrm{e}^{-t\mathcal{D}^2} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int_0^\beta d\tau\, \mathrm{e}^{-t(\mathcal{A}_n+\mathcal{B}_n)} \cdot 1\,, \tag{4.48}$$

where we have introduced the operators

$$\begin{aligned} \mathcal{A}_n &= \left(\frac{2\pi n}{\beta} + a_0\right)^2 + |\vec{a}|^2\,, \\ \mathcal{B}_n &= -i\partial_\tau \left(-i\partial_\tau + \frac{2\pi n}{\beta} + a_0\right)\,. \end{aligned} \tag{4.49}$$

The expression (4.48) is viewed as operating on a constant 1, and the derivatives only contribute when they encounter terms involving $|\vec{a}|^2$. Note that generally the position variables $\vec{a}^{\,b}(\tau)$ are only periodic up to a permutation of the identical D-particles, which ensures that the configuration of the coordinates is periodic. In the present case this means that the relative coordinate $\vec{a}(\tau)$ can be either periodic or anti-periodic. In both of these sectors, the distance $|\vec{a}(\tau)|$ is a periodic function, and hence so is the operator $\mathcal{A}_n$.

To unravel the expression (4.48), we use the generalization of the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^{-t(\mathcal{A}_n+\mathcal{B}_n)} = \mathrm{e}^{-t\mathcal{A}_n}\, \mathrm{e}^{\mathcal{C}_n} \mathrm{e}^{-t\mathcal{B}_n} \tag{4.50}$$

where

$$\mathcal{C}_n = -\frac{t^2}{2!}\, [\mathcal{A}_n, \mathcal{B}_n] - \frac{t^3}{3!} \Big([\mathcal{A}_n, [\mathcal{A}_n, \mathcal{B}_n]] + [[\mathcal{A}_n, \mathcal{B}_n], \mathcal{B}_n] \Big) + \cdots\,. \tag{4.51}$$

In the loop expansion we expand around the tree-level configuration whose equation of motion is $\ddot{\vec{a}} = 0$. The only non-vanishing time derivatives of the operator $\mathcal{A}_n$ are then $\dot{\mathcal{A}}_n = 2\vec{a}\cdot\dot{\vec{a}}$ and $\ddot{\mathcal{A}}_n = 2|\dot{\vec{a}}|^2$. Moreover, since the integrand of (4.48) is a periodic function, we can freely integrate by parts and drop surface terms. Using (4.50), (4.51) and the equations of motion, we may compute the functional determinants in (4.44) to second order in the expansion in time-derivatives of $|\vec{a}|$. After some tedious algebra, we arrive finally at

$$\begin{aligned}\mathrm{tr}_B \ln \mathcal{D}^2 = \int_0^\beta d\tau \Bigg[&\frac{1}{\beta}\ln\frac{1}{2}\Big(\cosh\beta|\vec{a}| - \cos\beta a_0\Big) + \frac{|\dot{\vec{a}}|^2}{96|\vec{a}|}\frac{\sinh\beta|\vec{a}|}{\cosh\beta|\vec{a}|-\cos\beta a_0}\times \\ &\times\left\{\frac{7}{|\vec{a}|^2} - \frac{7\beta}{|\vec{a}|\sinh\beta|\vec{a}|}\frac{1-\cosh\beta|\vec{a}|\cos\beta a_0}{\cosh\beta|\vec{a}|-\cos\beta a_0} + \right. \\ &\left. +\beta^2\frac{\cos\beta a_0(\cosh\beta|\vec{a}|+\cos\beta a_0)-2}{(\cosh\beta|\vec{a}|-\cos\beta a_0)^2}\right\} + \cdots\Bigg]. \end{aligned} \tag{4.52}$$

where the ellipses denote terms which are of third and higher orders in time-derivatives of $\mathcal{A}_n$. In arriving at (4.52) we have evaluated the sum $\sum_n \ln\mathcal{A}_n$ using the formula (4.13),[7] and the sums $\sum_n \mathcal{A}_n^{-k}$ may then be computed from the formula $(k-1)!\,\mathcal{A}_n^{-k} = -(-\partial/\partial|\vec{a}|^2)^k \times \ln\mathcal{A}_n$. There are eight contributions of the form (4.52), one for each of the directions transverse to the plane of motion. From this result we must also subtract the fermionic contribution which comes from evaluating the determinant with anti-periodic boundary conditions. The net effect of inserting the anti-periodic delta-function into (4.46) is to replace a_0 by $a_0 + \pi/\beta$ everywhere, i.e. $\cos\beta a_0 \to -\cos\beta a_0$ in (4.52).

The final quantity we need is the velocity corrected determinant (the first two determinants of (4.44)). This can be straightforwardly evaluated by using the description given in the previous subsection of how the oscillator frequencies are modified in the velocity-dependent potential between two D0-branes. The leading order term $8\ln\frac{\cosh\beta|\vec{a}|-\cos\beta a_0}{\cosh\beta|\vec{a}|+\cos\beta a_0}$ with no time derivatives is corrected at finite velocity to

$$\begin{aligned}&6\ln(\cosh\beta|\vec{a}|-\cos\beta a_0) + \ln\left(\cosh\beta\sqrt{|\vec{a}|^2+2i|\dot{\vec{a}}|}-\cos\beta a_0\right) + \\ &+\ln\left(\cosh\beta\sqrt{|\vec{a}|^2-2i|\dot{\vec{a}}|}-\cos\beta a_0\right) - 4\ln\left(\cosh\beta\sqrt{|\vec{a}|^2+i|\dot{\vec{a}}|}+\cos\beta a_0\right) - \\ &-4\ln\left(\cosh\beta\sqrt{|\vec{a}|^2-i|\dot{\vec{a}}|}+\cos\beta a_0\right). \end{aligned} \tag{4.53}$$

By summing the expansion of (4.53) to second order in the velocity $|\dot{\vec{a}}|$ and the eight order $|\dot{\vec{a}}|^2$ contributions of (4.52), and subtracting the eight order $|\dot{\vec{a}}|^2$ terms in (4.52) with

[7] The equality (4.13) is valid up to a function of x which is unity at all of the zeroes $x = -my$. This would produce different asymptotic behaviour in the complex plane. Here we have defined it so that it is odd under reflection of x but not periodic under the shift $x \mapsto x + y$. There is no way to preserve both of these symmetries, and this gives a simple example of an anomaly.

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$\cos\beta a_0 \to -\cos\beta a_0$, we arrive finally at the effective action

$$\begin{aligned}
S_{\rm eff} = \int_0^\beta d\tau \Biggl\{ & \frac{T_0}{4}\, v^2 + \frac{8}{\beta} \ln \frac{\cosh\frac{\beta r}{2\pi\alpha'} - \cos\beta a_0}{\cosh\frac{\beta r}{2\pi\alpha'} + \cos\beta a_0} + \frac{v^2}{12r} \frac{1}{\cosh^2\frac{\beta r}{2\pi\alpha'} - \cos^2\beta a_0} \times \\
& \times \Biggl[38 \frac{2\pi\alpha'}{r^2} \sinh\frac{\beta r}{2\pi\alpha'} \cos\beta a_0 - \\
& - \frac{38\beta\cosh\frac{\beta r}{2\pi\alpha'}\cos\beta a_0}{r} \frac{2-\cosh^2\frac{\beta r}{2\pi\alpha'} - \cos^2\beta a_0}{\cosh^2\frac{\beta r}{2\pi\alpha'} - \cos^2\beta a_0} + \\
& + \frac{2\beta^2}{2\pi\alpha'} \frac{\sinh\frac{\beta r}{2\pi\alpha'}\, \cos\beta a_0}{\left(\cosh^2\frac{\beta r}{2\pi\alpha'} - \cos^2\beta a_0\right)^2} \times \\
& \times \left(\cosh^4\frac{\beta r}{2\pi\alpha'} - \cos^2\beta a_0 \left(2-\cos^2\beta a_0\right) 6\cosh^2\frac{\beta r}{2\pi\alpha'} \sin^2\beta a_0 \right) \Biggr] + \cdots \Biggr\},
\end{aligned} \tag{4.54}$$

where the D0-brane separation r is time-dependent and obeys periodic boundary conditions on the temperature circle. The quantity $T_0/2$ is the reduced mass of the two D-particle system, while $r/2\pi\alpha'$ is the energy of a string which has Dirichlet boundary conditions on hypersurfaces a distance r apart. Note that the effective action is an odd function of the variable $x = \cos\beta a_0$. The action (4.54) simplifies in the limit $\beta r \gg 2\pi\alpha'$, and to leading orders in the low-temperature expansion we have

$$\begin{aligned}
S_{\rm eff} = \int_0^\beta d\tau \Biggl[& \frac{T_0}{4}\, v^2 - \frac{16}{\beta}\, {\rm e}^{-\beta r/2\pi\alpha'} \cos\beta a_0 + \\
& + \frac{v^2}{6r} \cos\beta a_0 {\rm e}^{-\beta r/2\pi\alpha'} \left(19\, \frac{2\pi\alpha'}{r^2} + \frac{19\beta}{r} + \frac{\beta^2}{2\pi\alpha'} \right) + \\
& + \mathcal{O}\left({\rm e}^{-3\beta r/2\pi\alpha'} \cos^3\beta a_0 \right) + \cdots \Biggr].
\end{aligned} \tag{4.55}$$

The second term in (4.54) has a direct interpretation in string perturbation theory. One can compute the annulus diagram for the open superstring, in compactified euclidean time of circumference β, whose ends lie on two stationary D0-branes separated by distance r. The charges at the endpoints of the string couple to a constant U(1) gauge field which is parametrized by $\nu \in (-1, 1]$, and which enters the problem through the quantized temporal momentum $p^0 = 2\pi(n-\nu)/\beta$, $n \in \mathbb{Z} + \frac{a-1}{2}$, of the open string whose worldsheet winds around the spacetime cylinder. Then, the one-loop thermal partition function of the string gas can be written as [28]

$$Z_{\rm str}(\beta, r, \nu) = \prod_{N=0}^\infty \left| \frac{1 + {\rm e}^{-\beta E_N + i\pi\nu}}{1 - {\rm e}^{-\beta E_N + i\pi\nu}} \right|^{2d_N}, \tag{4.56}$$

where the superstring spectrum is given by

$$\sqrt{\alpha'} E_N = \sqrt{\frac{r^2}{4\pi^2\alpha'} + N}\,, \tag{4.57}$$

with N the oscillator occupation number, and d_N is the degeneracy of superstring states at level N which may be computed from the generating function

$$8\prod_{n=1}^{\infty}\left(\frac{1+\mathrm{e}^{-nl}}{1-\mathrm{e}^{-nl}}\right)^{8}=\sum_{N=0}^{\infty}d_N\,\mathrm{e}^{-Nl}\,. \tag{4.58}$$

For the lowest level we have $d_0 = 8$ and $E_0 = r/2\pi\alpha'$. The factor of 2 in the power of (4.56) is again due to the exchange symmetry of the string endpoints. The partition function (4.56) is equal to the ratio of the Fermi and Bose distributions with power (twice) the degeneracy of states and the parameter $i\nu$ playing the role of a chemical potential. The static limit $v = 0$ of (4.54) coincides with $\ln Z_{\rm str}$ truncated to the massless modes ($N = 0$) with the identification $\pi\nu = \beta a_0$.

As stressed in [2], the integration over a_0 of the effective action is required for gauge invariance of the free energy, or equivalently to enforce Gauss' law for the charges at the ends of the open string which are induced on D-branes. The effective potential (4.43) between D0-branes is thereby given from (4.54) as $S_{\rm eff}[\vec y^{\,a}] = -\ln\int_{-1}^{1}d\nu\,\mathrm{e}^{-S_{\rm eff}}$. The reason has a natural explanation in the closed string formulation, obtained by mapping the open string annulus diagram onto the cylinder diagram via the standard modular transformation. Then, the path integral describes the closed string propagator corresponding to the interaction between two D0-branes, rather than the thermal partition function as in the case of an open string. When two D0-branes interact, they can exchange several closed strings, not only one. As all such exchanges are of the same order in the string coupling constant, they exponentiate since the closed strings are identical and naturally produce the result (4.56) in the closed string language. Furthermore, in this formulation it is clear that there is only a single gauge field parameter ν for each multi-string term, because now the system is composed of just two interacting D0-branes rather than a gas of D-particles. These facts result in the effective potential as claimed.

In the static limit, this potential is logarithmic and attractive at short distances. The singularity occurs as the D0-branes fall on top of one another, in which case the non-abelian gauge symmetry which is broken by separated branes is restored [58]. Then, the one-loop approximation breaks down, and this demonstrates that the thermodynamics of D0-branes must be treated as a problem in *quantum* statistical mechanics, defined by the path integral (4.42) over both periodic and anti-periodic trajectories $\vec y(\tau)$. On the other hand, the leading velocity corrections to the static thermal potential are repulsive at short distances. Note that these corrections are of order v^2 and vanish in the zero temperature limit, as expected. This illustrates that the moduli space of the two D-particle system is curved in a very non-trivial way by thermal effects. The calculation presented in this subsection can be extended to compute the thermal corrections to the order v^4 gravitational interaction between moving D-branes, and thereby to shed more light on the dynamical role of D0-branes in black hole thermodynamics.

5. Noncommutative D-brane geometries

A recent surge of activity in string theory and quantum field theory has come with the realization that D-branes in certain background supergravity fields lend an explicit realization to some old ideas that the classical notions of spacetime and general relativity at very short distances must be drastically altered. At these length scales quantum gravitational fluctuations cannot be ignored and spacetime is no longer described by a differentiable manifold. Following earlier suggestions, noncommutative geometry has been proposed as the appropriate mathematical framework to describe the short scale structure of spacetime, and in particular non-perturbative properties of string theory. The fact that quantum field theory on a noncommutative space arises naturally in string theory [51] and M-theory [20] suggests that spacetime noncommutativity is a general feature of a unified theory of quantum gravity.

D-brane worldvolumes become noncommutative manifolds when there is a constant Neveu-Schwarz two-form field $\mathcal{B}_{\mu\nu}$ on them. This field can be coupled to the usual open string σ-model (2.23) in the neutral limit $e_1 = e_2 = e$ by adding the topological action $-i\int_\Sigma x^*\mathcal{B}$. This term is a total derivative and so it only contributes to the boundary conditions on the string fields, not to their equations of motion. The endpoints of the string (the boundaries $\sigma = 0, 1$ of Σ) are now interpreted as ending on a D-brane of a certain dimensionality. The $\mathcal{B}$ field only appears in a gauge invariant combination with the U(1) gauge field on the brane as $B_{\mu\nu} = \mathcal{B}_{\mu\nu} - eF_{\mu\nu}$. Therefore, the uniform Neveu-Schwarz background field is equivalent to a constant electromagnetic field on the D-brane and so the following analysis will unify our discussions from earlier sections. Since the background fields can be gauged away in the directions transverse to the D-brane worldvolume, we shall only study the quantities associated with the worldvolume hyperplane itself.

When the target space has euclidean signature, the noncommutativity of the string endpoint coordinates can be understood through the analogy, discussed in section 2.2.2, between the external field problem for open strings and the classic Landau problem. The σ-model action describing the coupling of strings to a magnetic field on the branes is formally equivalent to the Landau action

$$S_{\rm L} = \int d\tau \left(\frac{M}{2}\,\dot{\vec{y}}^{\,2} - Q\,\dot{\vec{y}}\cdot\vec{A}\right), \tag{5.1}$$

which describes the motion of a particle of charge Q and mass M in the plane $\vec{y} = (y^1, y^2)$ and in the presence of a uniform perpendicular magnetic field of magnitude B. Here $A_i = -\frac{B}{2}\,\epsilon_{ij}\,y^j$ is the corresponding vector potential. In the limit of a strong magnetic field $B \gg M$ (with M fixed), the action (5.1) is already expressed in phase space with the spatial coordinates y^1, y^2 being the canonically conjugate variables. In canonical quantization, the position variables become noncommuting operators with $[y^i, y^j] = (i/QB)\,\epsilon^{ij}$. The mass gap between Landau levels is QB/M, so that the limit of strong magnetic field projects the quantum mechanical spectrum of this system onto the lowest Landau level and the spatial coordinates live in a noncommutative space. As we will see in the following, this is precisely what happens to the string endpoints when there is a constant magnetic field on the D-branes, and the D-brane worldvolume becomes a noncommutative manifold. However, as one can anticipate from our earlier analyses, the picture changes drastically in Minkowski signature corresponding to an electric field on the branes.

5.1 Magnetic fields and noncommutative field theory

We will start with the case of euclidean spacetime, so that $B_{\mu\nu}$ represents a uniform magnetic field on the D-branes, which we assume is of maximal rank. The open string boundary conditions are given by (2.24) with the replacements of $-e_k F_{\mu\nu}$ by $B_{\mu\nu}$ everywhere. To see how noncommutative geometry arises on the D-brane worldvolume, we will use the operatorial, covariant quantization formalism of section 2.2.1, but now in full generality and with a more careful analysis of the canonical quantization. The mode expansions which solve the bulk equations of motion $\Box\, x^\mu = 0$ and the boundary conditions (2.24) are given by the familiar expressions

$$
\begin{aligned}
x^\mu(\tau,\sigma) \;=\; & y^\mu 2\alpha' \left(G^{-1}\right)^{\mu\nu} \left(q_\nu\,\tau - 2\pi^2\alpha' B_\nu{}^\lambda\, q_\lambda\,\sigma\right) + \\
& + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{\mathrm{e}^{\,-in\tau}}{n} \left(i\, a_n^\mu \cos n\pi\sigma - 2\pi\alpha' B^\mu{}_\nu\, a_n^\nu \sin n\pi\sigma\right) ,
\end{aligned}
\tag{5.2}
$$

where

$$
G = \mathbb{1} - (2\pi\alpha' B)^2 \tag{5.3}
$$

is the usual Born-Infeld factor. As is evident from the expression for the string propagator in the background field [51], the symmetric tensor (5.3) is the *open* string metric, i.e. the metric seen by the endpoints of the string, while $\mathbb{1}$ is the bulk, closed string metric that defines the σ-model action.

We can now straightforwardly compute the equal-time, canonical commutation relations as described in section 2.2.1. Those involving the worldsheet momentum density uniquely fix the usual Heisenberg commutation relations for the zero modes x^μ, q_ν and the standard Heisenberg-Weyl commutation relations for the oscillatory modes a_n^μ in the metric G. The subtle relation comes from the equal-time commutator $[x^\mu(\tau,\sigma), x^\nu(\tau,\sigma')] = 0$ [4, 19]. By using the Heisenberg-Weyl commutation relations and the mode expansion (5.2), this commutator is readily seen to be given by

$$
\left[x^\mu(\tau,\sigma) x^\nu(\tau,\sigma')\right] = [y^\mu, y^\nu] - i\,\theta^{\mu\nu}\left(\sigma+\sigma'\right) - - i\,\theta^{\mu\nu} \sum_{n\neq 0} \frac{1}{n} \sin n\pi\left(\sigma+\sigma'\right) , \tag{5.4}
$$

where

$$
\theta = -(2\pi\alpha')^2\, G^{-1} B \tag{5.5}
$$

is the open string external field. By integrating the completeness relation (2.8) we may arrive at the Fourier series expansion

$$
\sum_{n\neq 0} \frac{1}{n} \sin n\pi\left(\sigma+\sigma'\right) = 1 - \left(\sigma+\sigma'\right) \tag{5.6}
$$

for $\sigma + \sigma' \in (0,2)$. From (5.4) we see that for $\sigma, \sigma' \in (0,1)$ in the *bulk* of the string worldsheet, the canonical commutation relations may be satisfied by fixing the commutators of the zero mode position operators to be

$$
[y^\mu, y^\nu] = i\,\theta^{\mu\nu} \,. \tag{5.7}
$$

The y^μ therefore generate a noncommutative algebra of operators and are interpreted as coordinates on a noncommutative space. They guarantee that the equal-time commutators are unmodified in the bulk of the worldsheet. This must be the case, since the coupling to the external field only modifies the boundaries of the string worldsheet, not the interior.

However, from (5.4) and (5.7) it now follows that the open string endpoint coordinates become noncommuting operators,

$$\begin{aligned}\Big[x^\mu(\tau,0)\,,\,x^\nu(\tau,0)\Big] &= i\,\theta^{\mu\nu}\,,\\ \Big[x^\mu(\tau,1)\,,\,x^\nu(\tau,1)\Big] &= -i\,\theta^{\mu\nu}\,,\end{aligned} \tag{5.8}$$

with all other embedding field commutators vanishing. The commutation relations (5.8) arise from the compatibility of the open string boundary conditions with the standard commutators, and they imply that the presence of the B-field deforms the D-brane worldvolume to a noncommutative manifold. Note that the noncommutativity of the worldvolume coordinates cannot be probed by any closed string objects (such as supergravity fields). This is because of the flip in sign between the commutators (5.8) at the two ends of the string which arise from the change of orientation. The left and right moving modes receive equal and opposite contributions from the B field and the noncommutativity averages out in the region between the two D-branes. In fact, one can explicitly calculate that the open string center of mass coordinates $x^\mu_{\rm cm}(\tau) = \int_0^1 d\sigma\, x^\mu(\tau,\sigma)$ commute. Thus the transverse space remains an ordinary (commutative) manifold. Indeed, we recall that the external field in the neutral case does not change the physical spectrum of the theory. The only effect of the magnetic field is the change the metric $\mathbb{1}$ to the open string metric (5.3). Recall also from section 2.2 that the open string is not point-like, but rather behaves like a neutral magnetic dipole whose two endpoints are at different positions. The dipole grows in the direction transverse to the motion by an amount proportional to $B_{\mu\nu}\,q^\nu$, and the fuzziness of space originates from its size [53, 11].

It is also possible to see noncommutativity in the charged string case [18], which formally corresponds to different external fields $B_k = \mathcal{B} - e_k F$ on the two D-branes between which the open strings stretch. By using the mode expansions (2.28) and the canonical commutation relations (2.30), (2.31) we find

$$\begin{aligned}\Big[x^+(\tau,\sigma)\,,\,x^-(\tau,\sigma')\Big] = \frac{1}{2\alpha'(B_1-B_2)} + \sum_{n=-\infty}^{\infty}\frac{1}{n-\alpha}\cos\Big((n-\alpha)\pi\sigma + \arctan 2\pi\alpha' B_1\Big)\,\times\\ \times\,\cos\Big((n-\alpha)\pi\sigma' + \arctan 2\pi\alpha' B_1\Big)\,.\end{aligned} \tag{5.9}$$

By using the identity

$$\sum_{n=1}^{\infty}\frac{2\alpha}{\alpha^2-n^2} + \frac{1}{\alpha} = \pi\cot\pi\alpha \tag{5.10}$$

for $\alpha\notin\mathbb{Z}$, we may infer the noncommutativity relations

$$\begin{aligned}\Big[x^+(\tau,0)\,,\,x^-(\tau,0)\Big] &= i\,\theta_1\\ \Big[x^+(\tau,1)\,,\,x^-(\tau,1)\Big] &= -i\,\theta_2\end{aligned} \tag{5.11}$$

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with all other embedding field commutators vanishing [18], where $\theta_k = (2\pi\alpha')^2 B_k[1 - (2\pi\alpha' B_k)^2]^{-1}$. Thus the noncommutativity is localized at the string endpoints and is determined by the field strengths on the D-branes. Exactly the same noncommutativity factors are obtained as if one quantized an individual open string ending on the same D-brane. This simply reflects the fact that noncommutativity is an intrinsic property of the brane worldvolume and not of the short-distance probe that is used. Notice also that the noncommutativity parameters are proportional to the string scale α' and thereby represent genuine stringy effects. The results are in fact exact to all orders in α' and the string coupling constant g_s, because noncommutativity is a short distance effect which doesn't care about the worldsheet topology. The loop corrections to the above results have been analysed in [35] with the same conclusions.

These same results can be reached by studying operator product expansions of open string vertex operators [49, 51]. In this analysis one can identify a particular regime of the string theory in which the vertex operator algebra reduces to a deformation of the ring of functions $f(y)$ on the D-brane worldvolume [36, 51]. It corresponds to taking the correlated limits $\alpha' \to 0$ (the field theory limit), $g_s \to 0$ (weakly coupled strings), and $B_{\mu\nu} \to \infty$ (strong magnetic field), with the quantities $(\alpha')^2 B_{\mu\nu}$ and $g_s\sqrt{\det \alpha' B}$ finite. The open string metric (5.3) is given by $G = -(2\pi\alpha' B)^2$ in this limit, since the closed string metric effectively scales out of the problem. Furthermore, from (5.2) it follows that the massive string modes are also scaled away from the endpoint zero modes. Thus all closed string states are completely decoupled from the problem, i.e. the gravitational modes are removed and an effective field theory remains. However, this is not a conventional field theory, because the noncommutativity parameter is also finite in this limit, $\theta = 1/B$. Indeed, because of (5.7), the resulting projection of the vertex operator algebra is not an ordinary function algebra, but rather that which is obtained by deforming the pointwise multiplication $f(y)g(y)$ of two functions to a product defined by a bi-differential operator of infinite order [36, 51]. This is is given by the classic Moyal star-product

$$f(y) \star g(y) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial y^\mu}\,\frac{\partial}{\partial y'^\nu}\right) f(y)g(y')\bigg|_{y'=y} \tag{5.12}$$

which is associative, but non-local and noncommutative. The commutation relations (5.7) may then be satisfied by replacing ordinary operator products with star-products of the coordinates y^μ. Note that in this decoupling limit the σ-model action (2.23) reduces to a sum of two quantum mechanical, boundary actions for the endpoint charges which are each formally equivalent to the Landau action (5.1) in the limit $B \to \infty$. This limit is thereby analogous to the projection onto the lowest-lying Landau level. The effects of noncommutativity from the string zero modes is emphasized in [33].

Proceeding as before, it can be shown that the effective field theory is given by a noncommutative generalization, obtained by replacing ordinary (commutative) products of fields with the Moyal product (5.12), of the Dirac-Born-Infeld action [38] on the D-brane worldvolume which describes non-linear electrodynamics on a fluctuating membrane [37]. This can be used to identify the effective open string coupling constant as [51]

$$G_s = g_s\sqrt{\det(\mathbb{1} + 2\pi\alpha' B)}\,. \tag{5.13}$$

After supersymmetrization, the low-energy effective field theory of noncommutative D-branes is noncommutative supersymmetric Yang-Mills theory with 16 supercharges (the number of supersymmetries preserved by the D-branes in the B field background) and spacetime metric $G_{\mu\nu}$. The Yang-Mills coupling constant in the decoupling limit described above is given by $g_{\rm YM}^2 \propto G_s = g_s \sqrt{\det 2\pi\alpha' B}$.

Quantum field theory on a noncommutative space appears to be the unique consistent deformation of ordinary quantum field theory. These theories exhibit a variety of novel effects which lead to new physics that are not encountered in conventional quantum field theories. Many of these effects have counterparts in string theory, and noncommutative field theories are believed to lie somewhere between ordinary field theory and string theory. For instance, one of the most important results is that infrared and ultraviolet effects do not decouple in a noncommutative field theory [44], which can be understood from the fact that the open string dipoles grow in size with their energy. The larger the momentum, the larger is the spatial extension of the object. Furthermore, noncommutative scalar field theories can contain stable soliton solutions even if their commutative counterparts don't [26], and these noncommutative solitons can be realized as D-branes in string field theory. Because of these striking features, intensive studies have been initiated which use noncommutative quantum field theory to study D-branes in the presence of a background magnetic field.

5.2 Electric fields and noncommutative open string theory

Let us now Wick rotate to Minkowski signature and consider a uniform electric field $E = |\vec{E}\,|$ on the branes, i.e. $B_{ij} = 0$ and $E_i = -iB_{0i} \neq 0$. Then $\theta^{0i} \neq 0$ and the D-brane worldvolume is space/time noncommutative. There are several reasons why one is interested in such a noncommutative theory. First of all, the lack of commutativity of time is in conflict with our current understanding of quantum mechanics, where time is not an operator but rather a parameter which labels the evolution of the system. Understanding space/time noncommutativity may therefore shed light on the role of time in string theory and quantum gravity. Secondly, the space/time commutator implies the uncertainty relation $\Delta y^0\, \Delta y^i \sim \theta^{0i}$ between time and space. This is simply the string space/time uncertainty principle that has been advocated as a generic property of string theory [59]. Finally, in the absence of external fields the effective supersymmetric Yang-Mills theory on the four-dimensional worldvolume of coincident D3-branes is known to possess an exact Montonen-Olive S-duality $g_{\rm YM} \leftrightarrow 1/g_{\rm YM}$. In the presence of a background electromagnetic field we expect this symmetry to act as an electric-magnetic duality exchanging electric and magnetic degrees of freedom. Naively then, we expect that the strong coupling dual of spatially noncommutative Yang-Mills theory in four dimensions to be a temporally noncommutative gauge theory.

This latter line of reasoning is, however, incorrect. Noncommutative quantum field theory with a noncommuting time direction is neither unitary nor causal. It suffers from severe acausal effects such as events which precede their causes and objects which grow instead of Lorentz contract as they are boosted. For example, the open string electric dipoles extend longitudinally by an amount proportional to $\vec{E}\cdot\vec{q}$. However, the string theory in a background electric field is, at least perturbatively, unitary and causal, as is evident in first quantization. Stringy effects eventually conspire to cancel the acausal effects that arise (for instance in the zero mode dipoles), and the model at the level of string theory is perfectly well-defined. Therefore, while the S-dual of the electric field problem for open strings is certainly the corresponding magnetic one, the noncommutative Yang-Mills theories cannot be related in such a manner. What has gone wrong is that the electric field problem does not possess a noncommutative field theory limit [52, 27]. Recall from the previous subsection that one of the decoupling limits involved making the external magnetic field arbitrarily large. In the electric case, the system destabilizes above the critical value E_c. This instability is now reflected in the singularities that arise at $E = E_c$ in the open string parameters (5.3), (5.5) and (5.13). It prevents the correlated limit of the previous subsection from being taken. The electric field cannot be scaled to infinity, and so one cannot reach the field theoretic limit $\alpha' \to 0$ in which all string oscillator modes decouple.

The key point though is that the effective tension of an open string stretched along the direction of the electric field is given by (3.10). One can take a limit in which $\alpha' \to 0$, and the theory is space/time noncommutative and decouples the bulk modes, including gravity, off the branes. However, the effective string scale $\alpha'_{\rm eff} = 1/2\pi T_{\rm eff}$ is finite in this limit and the effective theory will be a string theory, not a field theory. For this, we rotate so that the electric field lies along the 1-axis, and rescale the coordinates so that the diagonal elements of the closed string metric in the 0–1 plane are proportional to $[1 - (2\pi\alpha' E)^2]^{-1}$. We then take the limit whereby the electric field becomes critical, $2\pi\alpha' E \to 1$, and $\alpha' \to 0$ with $\alpha'_{\rm eff}$ fixed. For finite α', the open strings are effectively tensionless in the limit $E \to E_c$ and the open string metric (5.3) is finite. The closed string metric scales to infinity, and the Moyal phases are determined by the effective string scale as $\theta = 2\pi\alpha'_{\rm eff}$ and are therefore finite. Recall now that the neutral open string spectrum is unaltered by the electric field, and so the open string states are of finite mass. Thus we are left with an open string theory on the D-brane worldvolume which is a space/time noncommutative manifold. The fact that the noncommutativity scale is intrinsically tied to the string scale means that in order to make sense of a noncommutative space/time manifold, one needs to make precise the notion of an Einstein spacetime at the string scale.

The main property of this string theory is that, unlike ordinary string theory which requires closed string states for its consistency, it is completely decoupled from the bulk worldsheet states. To see this, suppose that a light open string state tries to escape to the bulk by turning into a closed string state (via a modular transformation). For this to occur, the stretched open string has to bend over in order for its endpoints to touch each other. Part of it will stretch against the electric field and will thereby become very heavy as $E \to E_c$. Thus the closed string modes become infinitely massive, and energetics prevent the open strings which live on the branes from turning into closed strings and propagating into the bulk. Note that, according to (5.13), this string theory is interacting provided we scale the closed string coupling $g_s \to \infty$. Therefore, these open strings describe a particular limit of strongly coupled closed strings in a critical electric field.

We conclude that in the low energy limit considered above, the effective theory includes interacting open strings on the D-branes together with decoupled free closed strings in the bulk region. The open string theory is decoupled from gravity, and the underlying spacetime on the D-branes is noncommutative. This theory is known as noncommutative open string theory [52]. In the case of D3-branes it is the strong coupling dual of supersymmetric noncommutative Yang-Mills theory in four dimensions [27]. The action involving these open strings is related to the action of ordinary open string theory by the replacement of all ordinary products of string fields with the appropriate noncommutative Moyal products (5.12). The thermal ensembles, and in particular the Hagedorn behaviour [30], of this string theory are particularly interesting since this theory does not contain closed strings and decouples from gravity. In the conventional superstring theory, which is difficult to study because of the thermodynamic instabilities that arise in gravitating systems, there is a first order phase transition below the Hagedorn temperature [6]. In the present case, one finds that, in the scaling limit and as the temperature is increased, a massless closed string state appears in the bulk at precisely the Hagedorn temperature (3.9) arising from the open string density of states. The Hagedorn transition in this case is a second order phase transition, and the high temperature phase involves long fundamental strings separating from the D-branes on which the noncommutative open string theory is defined [30].

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